NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 24, 2011

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 6 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.

- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **10** questions adding up to **30** questions in all.

- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.

- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).

- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.

- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.
  The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.
  The symbol I will denote the identity matrix of appropriate order.
  We denote by $GL_n(\mathbb{R})$ (respectively, $GL_n(\mathbb{C})$) the group (under matrix multiplication) of invertible $n \times n$ matrices with entries from $\mathbb{R}$ (respectively, $\mathbb{C}$) and by $SL_n(\mathbb{R})$ (respectively, $SL_n(\mathbb{C})$), the subgroup of matrices with determinant equal to unity. The trace of a square matrix A will be denoted tr(A) and the determinant by det(A).
  The derivative of a function f will be denoted by f'.
  All logarithms, unless specified otherwise, are to the base e.

- **Calculators are not allowed.**

Section 1: Algebra

1.1 Given that the sum of two of its roots is zero, solve the equation:

$$6x^4 - 3x^3 + 8x^2 - x + 2 \ = 0.$$

1.2 From the following subgroups of $GL_2(\mathbb{C})$, pick out those which are abelian:

a. the subgroup of invertible upper triangular matrices;

b. the subgroup S defined by

$$S \ = \ \left\{ \left[\begin{array}{cc} a & b \\ -b & a \end{array} \right] \ ; \ a, b \in \mathbb{R}, \ \text{and} \ |a|^2 + |b|^2 = 1 \right\}.$$

c. the subgroup U defined by

$$U \ = \ \left\{ \left[\begin{array}{cc} a & b \\ -\overline{b} & \overline{a} \end{array} \right] \ ; \ a, b \in \mathbb{C}, \ \text{and} \ |a|^2 + |b|^2 = 1 \right\}.$$

1.3 Let

$$S^3 \ = \ \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \ : \ \sum_{i=1}^{4} x_i^2 = 1 \right\}.$$

This can be identified with the set U of Question 1.2c above via the identification

$$a \ = \ x_1 + ix_2, \ b \ = \ x_3 + ix_4$$

and hence automatically acquires a group structure. Compute the inverse of the element $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ in this group.

1.4 Pick out the pairs which are conjugate to each other in the respective groups:

a.

$$\left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right] \ \text{and} \ \left[\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right] \ \text{in } GL_2(\mathbb{R});$$

b.

$$\left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right] \ \text{and} \ \left[\begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right] \ \text{in } SL_2(\mathbb{R});$$

c.

$$\left[\begin{array}{cc} 1 & 3 \\ 0 & 2 \end{array} \right] \ \text{and} \ \left[\begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right] \ \text{in } GL_2(\mathbb{R}).$$

1.5 Let $\mathcal{R}$ be a (commutative) ring (with identity). Let $\mathcal{I}$ and $\mathcal{J}$ be ideals in $\mathcal{R}$. Pick out the true statements:

a. $\mathcal{I} \cup \mathcal{J}$ is an ideal in $\mathcal{R}$;

b. $\mathcal{I} \cap \mathcal{J}$ is an ideal in $\mathcal{R}$;

c.

$$\mathcal{I} + \mathcal{J} \ = \ \{x + y \ : \ x \in \mathcal{I}, y \in \mathcal{J}\}$$

is an ideal in $\mathcal{R}$.

1.6 Pick out the rings which are integral domains:
a. $\mathbb{R}[x]$, the ring of all polynomials in one variable with real coeffcients;
b. $C^1[0,1]$, the ring of continuously differentiable real-valued functions on the interval $[0,1]$ (with respect to pointwise addition and pointwise multiplication);
c. $\mathbb{M}_n(\mathbb{R})$, the ring of all $n \times n$ matrices with real entries.

1.7 Let $W \subset \mathbb{R}^4$ be the subspace defined by

$$W = \{x \in \mathbb{R}^4 \ : \ Ax = 0\}$$

where

$$A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}.$$

Write down a basis for W.

1.8 let V be the space of all polynomials in one variable with real coefficients and of degree less than, or equal to, 3. Define the linear transformation

$$T(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = \alpha_0 + \alpha_1(1+x) + \alpha_2(1+x)^2 + \alpha_3(1+x)^3.$$

Write down the matrix of T with respect to the basis

$$\{1, 1+x, 1+x^2, 1+x^3\}.$$

1.9 Let A be a 2×2 matrix with real entries which is not a diagonal matrix and which satisfies $A^3 = I$. Pick out the true statements:
a. $\mathrm{tr}(A) = -1$;
b. A is diagonalizable over $\mathbb{R}$;
c. $\lambda = 1$ is an eigenvalue of A.

1.10 Let A be a symmetric $n \times n$ matrix with real entries, which is *positive semi-definite*, *i.e.* $x^T Ax \geq 0$ for every (column) vector x, where x^T denotes the (row) vector which is the transpose of x. Pick out the true statements:
a. the eigenvalues of A are all non-negative;
b. A is invertible;
c. the principal minor Δ_k of A (*i.e.* the determinant of the $k \times k$ matrix obtained from the first k rows and first k columns of A) is non-negative for each $1 \leq k \leq n$.

Section 2: Analysis

2.1 Evaluate:

$$\lim_{x\to 0}(1+3x^2)^{5\cot x+2\frac{\operatorname{cosec}x}{x}}.$$

2.2 Test the following series for convergence:

a.

$$\sum_{n=1}^{\infty}(\sqrt[3]{n^3+1}-n);$$

b.

$$\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\frac{7}{4.5.6}+\cdots$$

2.3 Find the sum of the infinite series:

$$\frac{1}{6}+\frac{5}{6.12}+\frac{5.8}{6.12.18}+\frac{5.8.11}{6.12.18.24}+\cdots$$

2.4 Let $[x]$ denote the largest integer less than, or equal to, $x \in \mathbb{R}$. Find the points of discontinuity (if any) of the following functions:
a. $f(x)=[x^2]\sin \pi x,\ x>0;$
b. $f(x)=[x]+(x-[x])^{[x]},\ x\geq 1/2.$

2.5 Pick out the uniformly continuous functions:
a. $f(x)=\cos x\cos\frac{\pi}{x},\ x\in]0,1[;$
b. $f(x)=\sin x\cos\frac{\pi}{x},\ x\in]0,1[;$
c. $f(x)=\sin^2 x,\ x\in [0,\infty[;$

2.6 Evaluate:

$$\sum_{k=1}^{n} ke^{kx},\ x\in\mathbb{R}\backslash\{0\}.$$

2.7 Let $f:\mathbb{R}\to\mathbb{R}$ be a function which is differentiable at $x=a$. Evaluate the following:

a.

$$\lim_{x\to a}\frac{a^n f(x)-x^n f(a)}{x-a};$$

b.

$$\lim_{n\to\infty} n\left[\sum_{j=1}^{k} f\left(a+\frac{j}{n}\right)-kf(a)\right].$$

2.8 Find the cube roots of $-i$.

2.9 Evaluate:

$$\int_{\Gamma}\frac{z+2}{z}\,dz$$

where Γ is the semi-circle $z=2e^{i\theta}$, θ varying from 0 to π.

2.10 Find the points z in the complex plane where $f'(z)$ exists and evaluate it at those points:
a. $f(z)=x^2+iy^2;$
b. $f(z)=z\mathcal{I}\mathrm{m}(z)$, where $\mathcal{I}\mathrm{m}(z)$ denotes the imaginary part of z.

Section 3: Geometry

3.1 Let BC be a fixed line segment of length d in the plane. Let A be a point which moves such that sum of the lengths AB and AC is a constant k. Find the maximum value of the area of the triangle ΔABC.

3.2 Let $A = (0, 1)$ and $B = (2, 0)$ in the plane. Let O be the origin and $C = (2, 1)$. Let P move on the segment OB and let Q move on the segment AC. Find the coordinates of P and Q for which the length of the path consisting of the segments AP, PQ and QB is least.

3.3 A regular $2N$-sided polygon of perimeter L has its vertices lying on a circle. Find the radius of the circle and the area of the polygon.

3.4 Let BC be a fixed line segment of length d and let S be a fixed point whose distance from the line BC is $2a$. A point A moves such that the altitude of the triangle ΔABC from the vertex A is equal to the length of the line segment AS. Find the minimum possible value of the area of the triangle ΔABC.

3.5 Pick out the bounded sets:
a. S is the set of all points in the plane such that the product of its distances from a fixed pair of orthogonal straight lines is a constant;
b. $S = \{(x, y) \ : \ 4x^2 - 2xy + y^2 = 1\}$;
c. $S \ = \ \left\{(x, y) \ ; \ x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1\right\}$.

3.6 A circle in the plane $\mathbb{R}^2$ centred at C and of unit radius moves without slipping on the positive x-axis with C moving in the upper half-plane. Write down the parametric equations of the locus of the point P on the circle which coincides with the origin at the initial position of the circle and the parameter θ being the angle through which the radius CP has turned from the initial vertical position.

3.7 What are the direction cosines of the line joining the point $(1, -8, -2)$ to the point $(3, -5, 4)$ in $\mathbb{R}^3$?

3.8 Find the equation of the plane passing through the point $(1, -2, 1)$ and which is perpendicular to the planes $3x+y+z-2 = 0$ and $x-2y+z+4 = 0$.

3.9 Find the equation of the plane containing the line

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$$

and which is perpendicular to the plane $x + 2y + z = 12$.

3.10 A moving plane passes through a fixed point (a, b, c) (which is not the origin) and meets the coordinate axes at the points A, B and C, all away from the origin O. Find the locus of the centre of the sphere passing through the points O, A, B and C.