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2013 – MA

Test Paper Code : MA

Time : 3 Hours Maximum Marks : 100

INSTRUCTIONS

1. This question-cum-answer booklet has **32** pages and has **30** questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.
2. Write your **Registration Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on Page No. **4**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used :
   (a) For each correct answer, you will be awarded **2 (Two)** marks.
   (b) For each wrong answer, you will be awarded **–0.5 (Negative 0.5)** mark.
   (c) Multiple answers to a question will be treated as a wrong answer .
   (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
   (e) Negative marks for objective part will be carried over to total mark.
5. Answer the fill in the blank type and descriptive type questions only in the **space provided after each question**. No negative marks for fill in the blank type questions.
6. Do not write more than one answer for the same question. In case you attempt a fill in the blank or a descriptive question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing last only will be evaluated.
7. All answers must be written in blue/black/ blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Clip board, log tables, slide rule, cellular phone and electronic gadgets in any form are NOT allowed. Non Programmable Calculator is allowed.**
11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.
12. Refer to special instructions/useful data on the reverse.

A

2013 – MA

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

REGISTRATION NUMBER						

Name :

Test Centre :

Do not write your Registration Number or Name anywhere else in this question-cum-answer booklet.

I have read all the instructions and shall abide by them.

Signature of the Candidate

I have verified the information filled by the Candidate above.

Signature of the Invigilator

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Special Instructions/ Useful Data

$\mathbb{R}$	: The set of all real numbers
$\mathbb{N}$	: The set of all positive integers
f'	: First derivative of a real function f of single variable
$\mathbb{Z}_p$	: $\{0,1,\ldots,p-1\}$ with addition and multiplication modulo p
S^o	: Interior of a set $S \subseteq \mathbb{R}$
$\overline{S}$	: Closure of a set $S \subseteq \mathbb{R}$

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IMPORTANT NOTE FOR CANDIDATES

- **Questions 1-10 (objective questions) carry *two* marks each, questions 11-20 (fill in the blank questions) carry *three* marks each and questions 21-30 (descriptive questions) carry *five* marks each.**
- **The marking scheme for the objective type question, is as follows:**
  (a) For each correct answer, you will be awarded **2 (Two)** marks.
  (b) For each wrong answer, you will be awarded **-0.5 (Negative 0.5)** mark.
  (c) Multiple answers to a question will be treated as a wrong answer.
  (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
  (e) Negative marks for objective part will be carried over to total marks.
- **There is no negative marking for fill in the blank questions.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 4 only.**

Objective Questions

Q.1 Let $A=\begin{pmatrix}1 & 1 & 1\\ 3 & -1 & 1\\ 1 & 5 & 3\end{pmatrix}$ and V be the vector space of all $X\in\mathbb{R}^3$ such that $AX=0$. Then $\dim(V)$ is

(A) 0 (B) 1 (C) 2 (D) 3

Q.2 The value of n for which the divergence of the function $\vec{F}=\frac{\vec{r}}{|\vec{r}|^n}$, $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $|\vec{r}|\neq 0$, vanishes is

(A) 1 (B) -1 (C) 3 (D) -3

Q.3 Let A and B be subsets of $\mathbb{R}$. Which of the following is **NOT** necessarily true?

(A) $(A\cap B)^o\subseteq A^o\cap B^o$ (B) $A^o\cup B^o\subseteq(A\cup B)^o$

(C) $\overline{A}\cup\overline{B}\subseteq\overline{A\cup B}$ (D) $\overline{A}\cap\overline{B}\subseteq\overline{A\cap B}$

Q.4 Let $[x]$ denote the greatest integer function of x. The value of α for which the function

$$f(x)=\begin{cases}\dfrac{\sin[-x^2]}{[-x^2]}, & x\neq 0\\ \alpha, & x=0\end{cases}$$

is continuous at $x=0$ is

(A) 0 (B) $\sin(-1)$ (C) $\sin 1$ (D) 1

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Q.5 Let the function $f(x)$ be defined by

$$f(x)=\begin{cases} e^x, & x \text{ is rational} \\ e^{1-x}, & x \text{ is irrational} \end{cases}$$

for x in (0, 1). Then

(A) f is continuous at every point in $(0, 1)$
(B) f is discontinuous at every point in $(0, 1)$
(C) f is discontinuous only at one point in $(0, 1)$
(D) f is continuous only at one point in $(0, 1)$

Q.6 The value of the integral

$$\iint_D \sqrt{x^2+y^2}\, dx\, dy, \quad D=\left\{ (x,y)\in \mathbb{R}^2 :\ x \le x^2+y^2 \le 2x \right\}$$

is

(A) 0 (B) $\frac{7}{9}$ (C) $\frac{14}{9}$ (D) $\frac{28}{9}$

Q.7 Let

$$x_n=\left(1-\frac{1}{3}\right)^2\left(1-\frac{1}{6}\right)^2\left(1-\frac{1}{10}\right)^2\cdots\left(1-\frac{1}{\frac{n(n+1)}{2}}\right)^2, \quad n\ge 2.$$

Then $\lim_{n\to\infty} x_n$ is

(A) $\frac{1}{3}$ (B) $\frac{1}{9}$ (C) $\frac{1}{81}$ (D) 0

Q.8 Let p be a prime number. Let G be the group of all 2×2 matrices over $\mathbb{Z}_p$ with determinant 1 under matrix multiplication. Then the order of G is

(A) $(p-1)p(p+1)$
(B) $p^2(p-1)$
(C) p^3
(D) $p^2(p-1)+p$

A

Q.9 Let V be the vector space of all 2×2 matrices over $\mathbb{R}$. Consider the subspaces

$$W_1 = \left\{\begin{pmatrix} a & -a \\ c & d \end{pmatrix} : a, c, d \in \mathbb{R}\right\} \text{ and } W_2 = \left\{\begin{pmatrix} a & b \\ -a & d \end{pmatrix} : a, b, d \in \mathbb{R}\right\}.$$

If $m = \dim(W_1 \cap W_2)$ and $n = \dim(W_1 + W_2)$, then the pair (m, n) is

(A) $(2, 3)$ (B) $(2, 4)$ (C) $(3, 4)$ (D) $(1, 3)$

Q.10 Let $\wp_n$ be the real vector space of all polynomials of degree at most n. Let $D : \wp_n \to \wp_{n-1}$ and $T : \wp_n \to \wp_{n+1}$ be the linear transformations defined by

$$D\left(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n\right) = a_1 + 2a_2x + \ldots + na_nx^{n-1},$$

$$T\left(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n\right) = a_0x + a_1x^2 + a_2x^3 + \ldots + a_nx^{n+1},$$

respectively. If A is the matrix representation of the transformation $DT - TD : \wp_n \to \wp_n$ with respect to the standard basis of $\wp_n$, then the trace of A is

(A) $-n$ (B) n (C) $n+1$ (D) $-(n+1)$

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Answer Table for Objective Questions

Write the Code of your chosen answer only in the 'Answer' column against each Question Number. Do not write anything else on this page.

Question Number	Answer	Do not write in this column
01		
02		
03		
04		
05		
06		
07		
08		
09		
10		

FOR EVALUATION ONLY

Number of Correct Answers		Marks	(+)
Number of Incorrect Answers		Marks	(−)
Total Marks in Question Nos. 1-10			( )

Fill in the blank questions

Q.11 The equation of the curve satisfying $\sin y \dfrac{dy}{dx} = \cos y (1 - x \cos y)$ and passing through the origin is

Ans:

Q.12 Let f be a continuously differentiable function such that $\int_0^{2x^2} f(t)\,dt = e^{\cos x^2}$ for all $x \in (0, \infty)$. The value of $f'(\pi)$ is

Ans:

Q.13 Let $u = \dfrac{y^2 - x^2}{x^2 y^2}$, $v = \dfrac{z^2 - y^2}{y^2 z^2}$ for $x \neq 0$, $y \neq 0$, $z \neq 0$. Let $w = f(u, v)$, where f is a real valued function defined on $\mathbb{R}^2$ having continuous first order partial derivatives. The value of $x^3 \dfrac{\partial w}{\partial x} + y^3 \dfrac{\partial w}{\partial y} + z^3 \dfrac{\partial w}{\partial z}$ at the point (1, 2, 3) is

Ans:

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Q.14 The set of points at which the function $f(x,y)=x^4+y^4-x^2-y^2+1,\ (x,y)\in\mathbb{R}^2$ attains local maximum is

Ans:

Q.15 Let C be the boundary of the region in the first quadrant bounded by $y=1-x^2$, $x=0$ and $y=0$, oriented counter-clockwise. The value of $\oint_C (xy^2dx-x^2y\,dy)$ is

Ans:

Q.16 Let $f(x)=\begin{cases}0, & -1\le x\le 0\\ x^4, & 0<x\le 1\end{cases}$. If $f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k+\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$ is the Taylor's formula for f about $x=0$ with maximum possible value of n, then the value of ξ for $0<x\le 1$ is

Ans:

Q.17 Let $\vec{F}=2z\hat{i}+4x\hat{j}+5y\hat{k}$, and let C be the curve of intersection of the plane $z=x+4$ and the cylinder $x^2+y^2=4$, oriented counter-clockwise. The value of $\oint_C \vec{F}\cdot d\vec{r}$ is

Ans:

A

Q.18 Let f and g be the functions from $\mathbb{R}\setminus\{0, 1\}$ to $\mathbb{R}$ defined by $f(x)=\frac{1}{x}$ and $g(x)=\frac{x-1}{x}$ for $x\in\mathbb{R}\setminus\{0, 1\}$. The smallest group of functions from $\mathbb{R}\setminus\{0, 1\}$ to $\mathbb{R}$ containing f and g under composition of functions is isomorphic to

Ans:

Q.19 The orthogonal trajectory of the family of curves $\frac{x^2}{2}+y^2=c$, which passes through (1, 1) is

Ans:

Q.20 The function to which the power series $\sum_{n=1}^{\infty}(-1)^{n+1}\, n\, x^{2n-2}$ converges is

Ans:

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Descriptive questions

Q.21 Let $0 < a \leq 1$, $s_1 = \frac{a}{2}$ and for $n \in \mathbb{N}$, let $s_{n+1} = \frac{1}{2}\left(s_n^2 + a\right)$. Show that the sequence $\{s_n\}$ is convergent, and find its limit.

Space for the answer

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Space for the answer

A

Q.22 Evaluate

$$\int_{1/4}^{1} \int_{\sqrt{x-x^2}}^{\sqrt{x}} \frac{x^2 - y^2}{x^2} \, dy \, dx$$

by changing the order of integration.

Space for the answer

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Space for the answer

A

Q.23 Find the general solution of the differential equation

$$x^2 \frac{d^3 y}{dx^3} + x \frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 6 \frac{y}{x} = \frac{x \ln x + 1}{x^2}, \quad x > 0.$$

Space for the answer

Space for the answer

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Q.24 Let S_1 be the hemisphere $x^2+y^2+z^2=1,\ z>0$ and S_2 be the closed disc $x^2+y^2\le 1$ in the xy plane. Using Gauss' divergence theorem, evaluate $\iint_S \bar{F}\cdot d\bar{S}$, where $\bar{F}=z^2x\hat{i}+\left(\frac{y^3}{3}+\tan z\right)\hat{j}+(x^2z+y^2)\hat{k}$ and $S=S_1\cup S_2$.

Also evaluate $\iint_{S_1} \bar{F}\cdot d\bar{S}$.

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Space for the answer

Q.25 Let

$$f(x,y)=\begin{cases}\dfrac{2\left(x^3+y^3\right)}{x^2+2y}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0).\end{cases}$$

Show that the first order partial derivatives of f with respect to x and y exist at $(0,0)$. Also show that f is not continuous at $(0,0)$.

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Space for the answer

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Q.26 Let A be an $n \times n$ diagonal matrix with characteristic polynomial $(x-a)^p(x-b)^q$, where a and b are distinct real numbers. Let V be the real vector space of all $n \times n$ matrices B such that $AB = BA$. Determine the dimension of V.

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Space for the answer

Q.27 Let A be an $n \times n$ real symmetric matrix with n distinct eigenvalues. Prove that there exists an orthogonal matrix P such that $AP = PD$, where D is a real diagonal matrix.

Space for the answer

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Space for the answer

Q.28 Let K be a compact subset of $\mathbb{R}$ with nonempty interior. Prove that K is of the form $[a, b]$ or of the form $[a, b] \setminus \bigcup I_n$, where $\{I_n\}$ is a countable disjoint family of open intervals with end points in K.

Space for the answer

Space for the answer

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Q.29 Let $f:[a, b] \to \mathbb{R}$ be a continuous function such that f is differentiable in (a,c) and (c, b), $a < c < b$. If $\lim\limits_{x \to c} f'(x)$ exists, then prove that f is differentiable at c and $f'(c) = \lim\limits_{x \to c} f'(x)$.

Space for the answer

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Space for the answer

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Q.30 Let G be a finite group, and let φ be an automorphism of G such that $\varphi(x) = x$ if and only if $x = e$, where e is the identity element in G. Prove that every $g \in G$ can be represented as $g = x^{-1}\varphi(x)$ for some $x \in G$. Moreover, if $\varphi(\varphi(x)) = x$ for every $x \in G$, then show that G is abelian.

Space for the answer

Space for the answer

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Space for rough work

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Space for rough work

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2013 - MA Objective Part (Question Number 1 – 10)	
Total Marks	Signature

Fill in the blanks Part and Descriptive Part					
Question Number	Marks		Question Number	Marks	
11			21		
12			22		
13			23		
14			24		
15			25		
16			26		
17			27		
18			28		
19			29		
20			30		
Total Marks in Fill in the blanks Part and Descriptive Part					

Total (Objective Part) :	
Total (Fill in the blanks Part and Descriptive Part) :	
Grand Total :	
Total Marks (in words) :	
Signature of Examiner(s) :	
Signature of Head Examiner(s) :	
Signature of Scrutinizer :	
Signature of Chief Scrutinizer :	
Signature of Coordinating Head Examiner :	