

**NATIONAL BOARD FOR HIGHER MATHEMATICS**

**M. A. and M.Sc. Scholarship Test**

**September 23, 2006**

**Time Allowed: 150 Minutes**

**Maximum Marks: 45**

**Please read, carefully, the instructions on the following page**

## INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 10 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **15** questions adding up to **45** questions in all.
- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- $\mathbb{N}$  denotes the set of natural numbers,  $\mathbb{Z}$  - the integers,  $\mathbb{Q}$  - the rationals,  $\mathbb{R}$  - the reals and  $\mathbb{C}$  - the field of complex numbers.  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space. The symbol  $]a, b[$  will stand for the open interval  $\{x \in \mathbb{R} \mid a < x < b\}$  while  $[a, b]$  will stand for the corresponding closed interval;  $[a, b[$  and  $]a, b]$  will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol  $I$  will denote the identity matrix of appropriate order.

## SECTION 1: ALGEBRA

**1.1** Compute  $(\sqrt{3} + i)^{14} + (\sqrt{3} - i)^{14}$  (Hint: Use De Moivre's theorem).

**1.2** Let  $p(x)$  be the polynomial  $x^3 - 11x^2 + ax - 36$ , where  $a$  is a real number. Assume that it has a positive root which is the product of the other two roots. Find the value of  $a$ .

**1.3** Identify which of the following groups (if any) is cyclic:

- (a)  $\mathbb{Z}_8 \oplus \mathbb{Z}_8$
- (b)  $\mathbb{Z}_8 \oplus \mathbb{Z}_9$
- (c)  $\mathbb{Z}_8 \oplus \mathbb{Z}_{10}$ .

**1.4** In each of the following examples determine the number of homomorphisms between the given groups:

- (a) from  $\mathbb{Z}$  to  $\mathbb{Z}_{10}$ ;
- (b) from  $\mathbb{Z}_{10}$  to  $\mathbb{Z}_{10}$ ;
- (c) from  $\mathbb{Z}_8$  to  $\mathbb{Z}_{10}$ .

**1.5** Let  $S_7$  be the group of permutations on 7 symbols. Does  $S_7$  contain an element of order 10? If the answer is "yes", then give an example.

**1.6** Let  $G$  be a finite group and  $H$  be a subgroup of  $G$ . Let  $O(G)$  and  $O(H)$  denote the orders of  $G$  and  $H$  respectively. Identify which of the following statements are necessarily true.

- (a) If  $O(G)/O(H)$  is a prime number then  $H$  is normal in  $G$ .
- (b) If  $O(G) = 2O(H)$  then  $H$  is normal in  $G$ .
- (c) If there exist normal subgroups  $A$  and  $B$  of  $G$  such that  $H = \{ab \mid a \in A, b \in B\}$  then  $H$  is normal in  $G$ .

**1.7** Which of the following statements are true?

- (a) There exists a finite field in which the additive group is not cyclic.
- (b) If  $F$  is a finite field, there exists a polynomial  $p$  over  $F$  such that  $p(x) \neq 0$  for all  $x \in F$ , where 0 denotes the zero in  $F$ .
- (c) Every finite field is isomorphic to a subfield of the field of complex numbers.

**1.8** Let  $V$  be a vector space of dimension 4 over the field  $\mathbb{Z}_3$  with 3 elements. What is the number of one-dimensional vector subspaces of  $V$ ?

**1.9** Let  $V$  be a vector space of dimension  $d < \infty$ , over  $\mathbb{R}$ . Let  $U$  be a vector subspace of  $V$ . Let  $S$  be a subset of  $V$ . Identify which of the following statements is true:

- (a) If  $S$  is a basis of  $V$  then  $U \cap S$  is a basis of  $U$ .
- (b) If  $U \cap S$  is a basis of  $U$  and  $\{s + U \in V/U \mid s \in S\}$  is a basis of  $V/U$  then  $S$  is a basis of  $V$ .
- (c) If  $S$  is a basis of  $U$  as well as  $V$  then the dimension of  $U$  is  $d$ .

**1.10** Let  $M(n, \mathbb{R})$  be the vector space of  $n \times n$  matrices with real entries. Let  $U$  be the subset of  $M(n, \mathbb{R})$  consisting  $\{(a_{ij}) \mid a_{11} + a_{22} + \dots + a_{nn} = 0\}$ . Is it true that  $U$  is a vector subspace of  $V$  over  $\mathbb{R}$ ? If so what is its dimension?

**1.11** Let  $A$  be a  $3 \times 3$  matrix with complex entries, whose eigenvalues are  $1, i$  and  $-2i$ . If  $A^{-1} = aA^2 + bA + cI$ , where  $I$  is the identity matrix, with  $a, b, c \in \mathbb{C}$ , what are the values of  $a, b$  and  $c$ ?

**1.12** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear transformation of  $\mathbb{R}^n$ , where  $n \geq 3$ , and let  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  be the eigenvalues of  $T$ . Which of the following statements are true?

- (a) If  $\lambda_i = 0$ , for some  $i = 1, \dots, n$ , then  $T$  is not surjective.
- (b) If  $T$  is injective, then  $\lambda_i = 1$  for some  $i$ ,  $1 \leq i \leq n$ .
- (c) If there is a 3-dimensional subspace  $U$  of  $V$  such that  $T(U) = U$ , then  $\lambda_i \in \mathbb{R}$  for some  $i$ ,  $1 \leq i \leq n$ .

**1.13** Let  $p(x) = a_0 + a_1x + \dots + a_nx^n$  be the characteristic polynomial of a  $n \times n$  matrix  $A$  with entries in  $\mathbb{R}$ . Then which of the following statements is true?

- (a)  $p(x)$  has no repeated roots.
- (b)  $p(x)$  can be expressed as a product of linear polynomials with real coefficients.
- (c) If  $p(x)$  can be expressed as a product of linear polynomials with real coefficients then there is a basis of  $\mathbb{R}^n$  consisting of eigenvectors of  $A$ .

**1.14** Let  $\mathbb{Z}_n$  be the ring of integers modulo  $n$ , where  $n$  is an integer  $\geq 2$ . Then complete the following:

- (a) If  $\mathbb{Z}_n$  is a field then  $n$  is ... .
- (b) If  $\mathbb{Z}_n$  is an integral domain then  $n$  is ... .
- (c) If there is an injective ring homomorphism of  $\mathbb{Z}_5$  to  $\mathbb{Z}_n$  then  $n$  is ... .

**1.15** Let  $C[0, 1]$  be the ring of continuous real-valued functions on  $[0, 1]$ , with addition and multiplication defined pointwise. For any subset  $S$  of  $C[0, 1]$  let  $Z(S) = \{x \in [0, 1] \mid f(x) = 0 \text{ for all } f \in S\}$ . Then which of the following statements are true?

- (a) If  $Z(S)$  is an ideal in  $C[0, 1]$  then  $S$  is closed in  $[0, 1]$ .
- (b) If  $Z(S)$  is a maximal ideal then  $S$  has only one point.
- (c) If  $S$  has only one point then  $Z(S)$  is a maximal ideal.

## SECTION 2: ANALYSIS

**2.1** Evaluate:

$$\lim_{\theta \rightarrow \frac{\pi}{2}} (1 - 5 \cot \theta)^{\tan \theta}.$$

**2.2** Evaluate:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^n \cos \left( \frac{\pi k}{2n} \right).$$

**2.3** Let  $x > 0$ . Define

$$f(x) = \int_0^x \frac{\sin xy}{y} dy.$$

Evaluate  $f'(x)$  as a function of  $x$ .

**2.4** What is the relation between the height  $h$  and the radius  $r$  of a right circular cylinder of fixed volume  $V$  and minimal total surface area?

**2.5** Find the coefficient of  $x^7$  in the Taylor series expansion of the function  $f(x) = \sin^{-1} x$  around 0 in the interval  $-1 < x < 1$ .

**2.6** Find the minimum value of the function:

$$f(x, y) = x^2 + 5y^2 - 6x + 10y + 6.$$

**2.7** Find the interval of convergence of the series:

$$(x+1) - \frac{(x+1)^2}{4} + \frac{(x+1)^3}{9} - \frac{(x+1)^4}{16} + \dots$$

**2.8** For what values of  $p$  does the following series converge?

$$1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots$$

**2.9** Pick out the series which are absolutely convergent:

(a)

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cos n\alpha}{n^2}$$

where  $\alpha \in \mathbb{R}$  is a fixed real number.

(b)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n \log n}{e^n}.$$

(c)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}.$$

**2.10** Pick out the functions which are continuous at least at one point in the real line:

(a)

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

(b)

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

(c)

$$f(x) = \begin{cases} \sin \pi x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

**2.11** Pick out the functions which are uniformly continuous:

(a)

$$f(x) = \frac{1}{x}, \quad x \in ]0, 1[.$$

(b)

$$f(x) = \frac{\sin x}{x}, \quad x \in ]0, 1[.$$

(c)

$$f(x) = \sin^2 x, \quad x \in \mathbb{R}.$$

**2.12** Let  $\mathcal{C}^1(\mathbb{R})$  denote the set of all continuously differentiable real valued functions defined on the real line. Define

$$A = \{f \in \mathcal{C}^1(\mathbb{R}) \mid f(0) = 0, f(1) = 1, |f'(x)| \leq 1/2 \text{ for all } x \in \mathbb{R}\}$$

where  $f'$  denotes the derivative of the function  $f$ . Pick out the true statement:

- (a)  $A$  is an empty set.
- (b)  $A$  is a finite and non-empty set.
- (c)  $A$  is an infinite set.

**2.13** Let  $\omega_i$ ,  $1 \leq i \leq 7$  denote the seventh roots of unity. Evaluate:

$$\prod_{i=1}^7 \omega_i.$$

**2.14** Pick out the true statements:

- (a)  $|\sin z| \leq 1$  for all  $z \in \mathbb{C}$ .
- (b)  $\sin^2 z + \cos^2 z = 1$  for all  $z \in \mathbb{C}$ .
- (c)  $\sin 3z = 3 \sin z - 4 \sin^3 z$  for all  $z \in \mathbb{C}$ .

**2.15** Evaluate:

$$\int_{\{|z|=2\}} \frac{dz}{(z-1)^3}.$$

### SECTION 3: GEOMETRY

**3.1** Write down the equation of the locus of a point which moves in the  $xy$ -plane so that it is equidistant from the straight lines  $y = x$  and  $y = -x$ .

**3.2** What is the shape of the locus of a point which moves in the plane so that it is equidistant from a given point  $A$  and a given straight line  $\ell$  (which does not contain the point  $A$ )?

**3.3** What is the area of a quadrilateral in the  $xy$ -plane whose vertices are  $(0, 0)$ ,  $(1, 0)$ ,  $(2, 3)$  and  $(0, 1)$ ?

**3.4** What is the surface area of the sphere whose equation is given by

$$x^2 + y^2 + z^2 - 4x + 6y - 2z + 13 = 0?$$

**3.5** What is the number of points of intersection of the curves

$$(x^2 + y^2 + 1)(x^2 + y^2 - 2x - 4y + 1) = 0 \quad \text{and} \quad x^2 + y^2 - 2x - 2y - 2 = 0?$$

**3.6** The plane  $m_1(x - 1) + m_2(y - 2) + m_3(z - 3) = 0$  is tangent to the surface  $x^3 + y^3 - z^3 + 3xyz = 0$  at the point  $(1, 2, 3)$ . What are the values of  $m_i$ ,  $1 \leq i \leq 3$  such that  $\sum_{i=1}^3 m_i^2 = 1$ ?

**3.7** Find the area enclosed by the circle formed by the intersection of the sphere  $x^2 + y^2 + z^2 - 2x - 2y - 2z = 1$  and the plane  $x + y + z = 1$ .

**3.8** Find the lengths of the semi-axes of the ellipse

$$2x^2 + 2xy + 2y^2 = 1.$$

**3.9** Let  $A, B, C$  and  $D$  be the vertices (in clockwise order) of a rectangle in the  $xy$ -plane. Let  $f(x, y) = ax + by$  for some fixed real numbers  $a$  and  $b$ . Given that  $f(A) = 5$ ,  $f(B) = f(D) = 10$ , find  $f(C)$  (here, if a point  $P = (u, v)$ , we write  $f(P)$  for  $f(u, v)$ ).

**3.10** Let  $A_i = (x_i, y_i), 1 \leq i \leq 3$  be the vertices of a triangle in the  $xy$ -plane. Then, given any point  $P = (x, y)$  inside the triangle, we can find three numbers  $\lambda_i = \lambda_i(x, y), 1 \leq i \leq 3$  such that

$$\begin{aligned} 0 \leq \lambda_i \leq 1, & \text{ for all } 1 \leq i \leq 3, \\ \lambda_1 + \lambda_2 + \lambda_3 &= 1, \\ x = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 & \text{ and } y = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3. \end{aligned}$$

If  $A_1 = (0, 0), A_2 = (1, 0)$  and  $A_3 = (0, 1)$ , write down  $\lambda_i, 1 \leq i \leq 3$  as functions of  $x$  and  $y$ .

**3.11** Consider the points  $A = (0, 2)$  and  $B = (1, 1)$  in the  $xy$ -plane. Consider all possible paths  $APB$  where  $P$  is an arbitrary point on the  $x$ -axis and  $AP$  and  $PB$  are straight line segments. Find the coordinates of the point  $P$  such that the length of the path  $APB$  is shortest amongst all such possible paths.

**3.12** What curve does the following equation represent in polar coordinates:

$$\frac{2}{r} = 1 + \frac{1}{2} \cos \theta?$$

**3.13** Find the angle between the planes  $2x - y + z = 6$  and  $x + y + 2z = 3$ .

**3.14** Which of the following equations represent bounded sets in the  $xy$ -plane?

- (a)  $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ .
- (b)  $xy = 1$
- (c)  $17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$ .

**3.15** Let  $P_n$  be a regular polygon of  $n$  sides inscribed in a circle of radius  $a$ , where  $a > 0$ . Let  $L_n$  and  $A_n$  be the perimeter and area of  $P_n$  respectively. Evaluate:

$$\lim_{n \rightarrow \infty} \frac{L_n^2}{A_n}.$$