NATIONAL BOARD FOR HIGHER MATHEMATICS

M. A. and M.Sc. Scholarship Test

September 19, 2009

Time Allowed: 150 Minutes

Maximum Marks: 30

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this question paper booklet contains 8 numbered (and printed) pages. The reverse of each printed page is blank and can be used for rough work.
- There are three parts to this test: Algebra, Analysis and Geometry. Each part consists of **10** questions adding up to **30** questions in all.
- Answer each question, as directed, in the space provided for it in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so do not answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a),(b) and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. There will be no partial credit.
- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order. All logarithms, unless specified otherwise, are to the base e.
- **Calculators are not allowed.**

Section 1: Algebra

1.1 A polynomial in the variable x leaves a remainder 2 when divided by $(x-3)$ and a remainder 3 when divided by $(x-2)$. Find the remainder when it is divided by $(x-2)(x-3)$.

1.2 Let $a_1, \cdots, a_n \in \mathbb{R}$. Evaluate the determinant:

$$\begin{vmatrix} 1+a_1 & a_2 & \cdots & a_n \\ a_1 & 1+a_2 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & \cdots & 1+a_n \end{vmatrix}.$$

1.3 Find the roots of the equation

$$27x^3 + 42x^2 - 28x - 8 \;=\; 0$$

given that they are in geometric progression.

1.4 Which of the following form a group?
a.

$$G \;=\; \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \;:\; a \in \mathbb{R},\; a \neq 0 \right\}$$

with respect to matrix multiplication.
b. $\mathbb{Z}_4$, the set of all integers modulo 4, with respect to multiplication.
c.

$$G \;=\; \{f : [0,1] \to \mathbb{R} \;;\; f \text{continuous}\}$$

with respect to the operation defined by $(f.g)(x) = f(x)g(x)$ for all $x \in [0,1]$.

1.5 Let G be a group and let H and K be subgroups of order 8 and 15 respectively. What is the order of the subgroup $H \cap K$?

1.6 Let G be a finite abelian group of odd order. Which of the following define an automorphism of G?
a. The map $x \mapsto x^{-1}$ for all $x \in G$.
b. The map $x \mapsto x^2$ for all $x \in G$.
c. The map $x \mapsto x^{-2}$ for all $x \in G$.

1.7 An algebraic number is one which occurs as the root of a monic polynomial with rational coefficients. Which of the following numbers are algebraic?
a. $5+\sqrt{3}$
b. $7+2^{\frac{1}{3}}$
c. $\cos\frac{2\pi}{n}$, where $n \in \mathbb{N}$

1.8 Given that the matrix
$$\begin{pmatrix} \alpha & 1 \\ 2 & 3 \end{pmatrix}$$
has 1 as an eigenvalue, compute its trace and its determinant.

1.9 Let A be a non-diagonal 2×2 matrix with complex entries such that $A = A^{-1}$. Write down its characteristic and minimal polynomials.

1.10 Pick out the true statements:
a. Let A and B be two arbitrary $n \times n$ matrices. Then
$$(A+B)^2 \;=\; A^2 + 2AB + B^2.$$
b. There exist $n \times n$ matrices A and B such that
$$AB - BA \;=\; I.$$
c. Let A and B be two arbitrary $n \times n$ matrices. If B is invertible, then
$$\mathrm{tr}(A) \;=\; \mathrm{tr}(B^{-1}AB)$$
where $\mathrm{tr}(M)$ denotes the trace of an $n \times n$ matrix M.

Section 2: Analysis

2.1 Evaluate:

$$\lim_{x\to\infty} x\left(\log\left(1+\frac{x}{2}\right)-\log\frac{x}{2}\right).$$

2.2 Evaluate:

$$\int_0^{\frac{\pi}{2}} \log\tan\theta \; d\theta.$$

2.3 Test the following series for convergence:
a.

$$\sum_{n=1}^{\infty}\left(n^{\frac{1}{n}}-1\right)^n$$

b.

$$\sum_{n=1}^{\infty}\frac{\sqrt{a_n}}{n}$$

where $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms.

2.4 Which of the following functions are continuous?
a.

$$f(x) \;=\; [x]+(x-[x])^{[x]},\; x\geq\frac{1}{2},$$

where $[x]$ denotes the largest integer less than, or equal to, x.
b.

$$f(x) \;=\; \lim_{n\to\infty}\frac{x^2e^{nx}+x}{e^{nx}+1},\; x\in\mathbb{R}.$$

c.

$$f(x) \;=\; \lim_{n\to\infty}\frac{1}{n}\log\left(e^n+x^n\right),\; x\geq 0.$$

2.5 Write down the coefficient of x^7 in the Taylor series expansion of the function

$$f(x) \;=\; \log(x+\sqrt{1+x^2})$$

about the origin.

2.6 Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable at $x=a$. Evaluate:

$$\lim_{n\to\infty}\left\{f\left(a+\frac{1}{n^2}\right)+f\left(a+\frac{2}{n^2}\right)+\cdots+f\left(a+\frac{n}{n^2}\right)-nf(a)\right\}.$$

2.7 Let

$$f(x,y) \;=\; x^4 - 2x^2y^2 + y^4 + x^2 - 6xy + 9y^2.$$

Examine whether f admits a local maximum or minimum at $(0,0)$.

2.8 Let $z = x + iy \in \mathbb{C}$ and let f be defined by

$$f(z) \;=\; y - x - 3x^2 i.$$

If C is the straightline segment joining $z = 0$ to $z = 1 + i$, compute

$$\int_C f(z)\, dz.$$

2.9 Let C be the contour consisting of the lines $x = \pm 2$ and $y = \pm 2$, described counterclockwise in the plane. Compute

$$\int_C \frac{z}{2z+1}\, dz.$$

2.10 Let

$$f(z) \;=\; \frac{5z-2}{z(z-1)}.$$

Write down the residues of f at each of its poles.

Section 3: Geometry

3.1 Let M_1 ad M_2 be two points in the plane whose polar coordinates are given as $(12, 4\pi/9)$ and $(12, -2\pi/9)$ respectively. Find the polar coordinates of the midpoint of the line segment joining these points.

3.2 The length of the sides of a rhombus is given as $5\sqrt{2}$. If two of its opposite vertices have coordinates $(3, -4)$ and $(1, 2)$, find the length of the altitude of the rhombus.

3.3 What figure does the equation

$$\sum_{i,j=1}^{2} a_{ij} x_i x_j = 1$$

represent when $A = (a_{ij})$ is the matrix given by
a.

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}?$$

b.

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}?$$

3.4 Find the locus of a point which moves such that the ratio of its distance from the point $(-5, 0)$ to its distance from the line $5x + 16 = 0$ is $5/4$.

3.5 Express the equations of the curves given below in parametric form in the form $f(x, y) = 0$.
a.

$$x = \frac{a}{2}\left(t + \frac{1}{t}\right), \ y = \frac{b}{2}\left(t - \frac{1}{t}\right).$$

b.

$$x = 2R\cos^2 t, \ y = R\sin 2t.$$

3.6 Let A_n be the area of the polygon whose vertices are given by the n-th roots of unity in the complex plane. Evaluate:

$$\lim_{n\to\infty} A_n.$$

3.7 Write down the equation of the diameter of the sphere

$$x^2 + y^2 + z^2 + 2x - 6y + z - 11 \;=\; 0$$

which is perpendicular to the plane $5x - y + 2z = 17$.

3.8 Find the values of a for which the plane $x + y + z = a$ is tangent to the sphere $x^2 + y^2 + z^2 = 12$.

3.9 Let $d(P, Q)$ denote the distance between two points P and Q in the plane. Let

$$A \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; xy = 0\}, \text{ and } B \;=\; \{(x, y) \in \mathbb{R}^2 \;:\; xy = 1\}.$$

Compute:

$$\inf_{p \in A, Q \in B} d(P, Q).$$

3.10 A ray, having the origin as its end-point, initially coincides with the x-axis and rotates about the origin in the plane with constant angular velocity ω. A point starts at the origin and moves along the ray with constant velocity v. Write down the parametric equations of the locus of the point in the form $x = \varphi(t), y = \psi(t)$, with time t as the parameter.