NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

January 27, 2007

Time Allowed: Two Hours

Maximum Marks: 40

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 12 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.

- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.

- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.

- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), and (c)).

- Points will be awarded in the above questions only if **all** the correct choices are made. **There will be no partial credit**.

- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, which is assumed to be endowed with its 'usual' topology. The symbol $]a,b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a,b]$ will stand for the corresponding closed interval; $[a,b[$ and $]a,b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order. The space of continuous real valued functions on an interval $[a,b]$ is denoted by $\mathcal{C}[a,b]$ and is endowed with its usual 'sup' norm. The space of continuously differentiable real valued functions on $[a,b]$ is denoted by $\mathcal{C}^1[a,b]$ and its usual norm is the maximum of the sup-norms of the function and its derivative.

Section 1: ALGEBRA

1.1 Let G be a group of order n. Which of the following conditions imply that G is abelian?
a. $n = 15$.
b. $n = 21$.
c. $n = 36$.

1.2 Which of the following subgroups are necessarily normal subgroups?
a. The kernel of a group homomorphism.
b. The center of a group.
c. The subgroup consisting of all matrices with positive determinant in the group of all invertible $n \times n$ matrices with real entries (under matrix multiplication).

1.3 List all the units in the ring of Gaussian integers.

1.4 List all possible values occuring as deg$f$ (degree of f) where f is an irreducible polynomial in $\mathbb{R}[x]$.

1.5 Write down an irreducible polynomial of degree 3 over the field $\mathbb{F}_3$ of three elements.

1.6 Let $A = (a_{ij})$ be an $n \times n$ matrix with real entries. Let A_{ij} be the cofactor of the entry a_{ij} of A. Let $\widetilde{A} = (A_{ij})$ be the matrix of cofactors. What is the rank of $\widetilde{A}$ under the following conditions:
(a) the rank of A is n?
(b) the rank of A is $\leq n - 2$?

1.7 Let A be an $n \times n$ matrix with complex entries which is not a diagonal matrix. Pick out the cases when A is diagonalizable.
a. A is idempotent.
b. A is nilpotent.
c. A is unitary.

1.8 For $n \geq 2$, let $\mathcal{M}(n)$ denote the ring of all $n \times n$ matrices with real entries. Which of the following statements are true?
a. If $A \in \mathcal{M}(2)$ is nilpotent and non-zero, then there exists a matrix $B \in \mathcal{M}(2)$ such that $B^2 = A$.
b. If $A \in \mathcal{M}(n)$, $n \geq 2$, is symmetric and positive definite, then there exists a symmetric matrix $B \in \mathcal{M}(n)$ such that $B^2 = A$.
c. If $A \in \mathcal{M}(n)$, $n \geq 2$, is symmetric, then there exists a symmetric matrix $B \in \mathcal{M}(n)$ such that $B^3 = A$.

1.9 Which of the following matrices are non-singular?
a. $I + A$ where $A \neq 0$ is a skew-symmetric real $n \times n$ matrix, $n \geq 2$.
b. Every skew-symmetric non-zero real 5×5 matrix.
c. Every skew-symmetric non-zero real 2×2 matrix.

1.10 Let V be a real finite-dimensional vector space and f and g non-zero linear functionals on V. Assume that $\ker(f) \subset \ker(g)$. Pick out the true statements.
a. $\ker(f) = \ker(g)$.
b. $\ker(g)/\ker(f) \cong \mathbb{R}^k$ for some k such that $1 \leq k < n$.
c. There exists a constant $c \neq 0$ such that $g = cf$.

Section 2: ANALYSIS

2.1 In each of the following cases, state whether the series is absolutely convergent, conditionally convergent (*i.e.* convergent but not absolutely convergent) or divergent.

a.

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{2n+3}.$$

b.

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+2}.$$

c.

$$\sum_{n=1}^{\infty}(-1)^n \frac{n\log n}{e^n}.$$

2.2 Determine the interval of convergence of the series:

$$\sum_{n=1}^{\infty}(-1)^{n-1} \frac{(x-1)^n}{n}.$$

2.3 What is the cardinality of the following set?

$$A \;=\; \{f \in \mathcal{C}^1[0,1] \;:\; f(0)=0, f(1)=1, |f'(t)| \le 1 \;\text{ for all }\; t \in [0,1]\}.$$

2.4 Evaluate:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\left[\frac{n}{2}\right]} \cos\left(\frac{k\pi}{n}\right)$$

where $\left[\frac{n}{2}\right]$ denotes the largest integer not exceeding $\frac{n}{2}$.

2.5 Which of the following improper integrals are convergent?

a.

$$\int_1^{\infty} \frac{dx}{\sqrt{x^3+2x+2}}.$$

b.

$$\int_0^{5} \frac{dx}{x^2-5x+6}.$$

c.

$$\int_0^{5} \frac{dx}{\sqrt[3]{7x+2x^4}}.$$

2.6 Which of the following series converge uniformly?

a.

$$\sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

over the interval $[-\pi, \pi]$ where $\sum_n |a_n| < \infty$ and $\sum_n |b_n| < \infty$.

b.

$$\sum_{n=0}^{\infty} e^{-nx} \cos nx$$

over the interval $]0, \infty[$.

c.

$$x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots$$

over the interval $[-1, 1]$.

2.7 Let $f \in \mathcal{C}[0, \pi]$. Determine the cases where the given condition implies that $f \equiv 0$.

a.

$$\int_0^{\pi} x^n f(x)\, dx \;=\; 0$$

for all integers $n \geq 0$.

b.

$$\int_0^{\pi} f(x) \cos nx\, dx \;=\; 0$$

for all integers $n \geq 0$.

c.

$$\int_0^{\pi} f(x) \sin nx\, dx \;=\; 0$$

for all integers $n \geq 1$.

2.8 Let C be the circle defined by $|z| = 3$ in the complex plane, described in the anticlockwise direction. Evaluate:

$$\int_C \frac{2z^2 - z - 2}{z - 2}\, dz.$$

2.9 Pick out the true statements:
a. Let f and g be analytic in the disc $|z| < 2$ and let $f = g$ on the interval $[-1, 1]$. Then $f \equiv g$.
b. If f is a non-constant polynomial with complex coefficients, then it can be factorized into (not necessarily distinct) linear factors.
c. There exists a non-constant analytic function in the disc $|z| < 1$ which assumes only real values.

2.10 Let $\Omega \subset \mathbb{C}$ be an open and connected set and let $f : \Omega \to \mathbb{C}$ be an analytic function. Pick out the true statements:
a. f is bounded if Ω is bounded.
b. f is bounded only if Ω is bounded.
c. f is bounded if, and only if, Ω is bounded.

Section 3: TOPOLOGY

3.1 In each of the following, f is assumed to be continuous. Pick out the cases when f *cannot* be onto.
a. $f : [-1, 1] \to \mathbb{R}$.
b. $f : [-1, 1] \to \mathbb{Q} \cap [-1, 1]$.
c. $f : \mathbb{R} \to [-1, 1]$.

3.2 Consider the set of all $n \times n$ matrices with real entries identified with $\mathbb{R}^{n^2}$, endowed with its usual topology. Pick out the true statements.
a. The subset of all invertible matrices is connected.
b. The subset of all invertible matrices is dense.
c. The subset of all orthogonal matrices is compact.

3.3 Pick out the functions that are uniformly continuous on the given domain.
a. $f(x) = \frac{1}{x}$ on the interval $]0, 1[$.
b. $f(x) = x^2$ on $\mathbb{R}$.
c. $f(x) = \sin^2 x$ on $\mathbb{R}$.

3.4 Let (X, d) be a metric space and let A and B be subsets of X. Define

$$d(A, B) \;=\; \inf\{d(a, b) \;:\; a \in A, b \in B\}.$$

Pick out the true statements.
a. If A and B are disjoint, then $d(A, B) > 0$.
b. If A and B are closed and disjoint, then $d(A, B) > 0$.
c. If A and B are compact and disjoint, then $d(A, B) > 0$.

3.5 Pick out the sets that are homeomorphic to the set

$$\{(x, y) \in \mathbb{R}^2 \;:\; xy = 1\}.$$

a. $\{(x, y) \in \mathbb{R}^2 \;:\; xy = 0\}$.
b. $\{(x, y) \in \mathbb{R}^2 \;:\; x^2 - y^2 = 1\}$.
c. $\{(x, y) \in \mathbb{R}^2 \;:\; x^2 + y^2 = 1\}$.

3.6 Let (X_i, d_i), $i = 1, 2, 3$, be the metric spaces where $X_1 = X_2 = X_3 = \mathcal{C}[0, 1]$ and

$$\begin{array}{rcl} d_1(f, g) & = & \sup_{t\in[0,1]} |f(x) - g(x)| \\ d_2(f, g) & = & \int_0^1 |f(x) - g(x)|\ dx \\ d_3(f, g) & = & (\int_0^1 |f(x) - g(x)|^2\ dx)^{\frac{1}{2}}. \end{array}$$

Let id be the identity map of $\mathcal{C}[0, 1]$ onto itself. Pick out the true statements.
a. $id : X_1 \to X_2$ is continuous.
b. $id : X_2 \to X_1$ is continuous.
c. $id : X_3 \to X_2$ is continuous.

3.7 Pick out the compact sets.
a. $\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C}\ :\ z_1^2 + z_2^2 = 1\}$.
b. The unit sphere in ℓ_2, the space of all square summable real sequences, with its usual metric

$$d(\{x_i\}, \{y_i\}) \ = \ \left(\sum_{i=1}^{\infty} |x_i - y_i|^2\right)^{\frac{1}{2}}.$$

c. The closure of the unit ball of $\mathcal{C}^1[0, 1]$ in $\mathcal{C}[0, 1]$.

3.8 Let $f : S^1 \to \mathbb{R}$ be any continuous map, where S^1 is the unit circle in the plane. Let

$$A \ = \ \{(x, y) \in S^1 \times S^1\ :\ x \neq y, f(x) = f(y)\}.$$

Is A non-empty? If the answer is ‘yes’, is it finite, countable or uncountable?

3.9 Let $f : S^1 \to \mathbb{R}$ be any continuous map, where S^1 is the unit circle in the plane. Let

$$A \ = \ \{(x, y) \in S^1 \times S^1\ :\ x = -y, f(x) = f(y)\}.$$

Is A non-empty?

3.10 Let $f \in \mathcal{C}^1[-1, 1]$ such that $|f(t)| \leq 1$ and $|f'(t)| \leq \frac{1}{2}$ for all $t \in [-1, 1]$. Let

$$A \ = \ \{t \in [-1, 1]\ :\ f(t) = t\}.$$

Is A non-empty? If the answer is ‘yes’, what is its cardinality?

Section 4: APPLIED MATHEMATICS

4.1 Let u be a smooth function defined on the ball centered at the origin and of radius $a > 0$ in $\mathbb{R}^3$. Assume that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \; = \; 1$$

throughout the ball. Compute:

$$\int_S \frac{\partial u}{\partial n} \, dS$$

where S is the sphere with centre at the origin and radius a and $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of u on S.

4.2 Consider a homogeneous fluid moving with velocity u in space. Write down the equation which expresses the principle of conservation of mass.

4.3 Let C be the equatorial circle on the unit sphere in $\mathbb{R}^3$ and let τ be the unit tangent vector to C taken in the anticlockwise sense. Compute:

$$\int_C \mathbf{F}.\tau \; ds$$

where $\mathbf{F}(x, y, z) = x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$.

4.4 Determine the value of the least possible positive number λ such that the following problem has a non-trivial solution:

$$\begin{aligned} u''(x) + \lambda u(x) \; &= \; 0, \; 0 < x < 1 \\ u'(0) \; = \; u'(1) \; &= \; 0. \end{aligned}$$

4.5 A pendulum of mass m and length ℓ is pulled to an angle α from the vertical and released from rest. Write down the differential equation satisfied by the angle $\theta(t)$ made by the pendulum with the vertical at time t, using the principle of conservation of energy.(If s is the arc length measured from the vertical position, then the velocity v is given by $v = \frac{ds}{dt}$.)

4.6 Find d'Alembert's solution to the problem:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} \; &= \; \frac{\partial^2 u}{\partial x^2} \quad x \in \mathbb{R}, \; t > 0 \\ u(x, 0) \; &= \; x^2 \quad\;\; x \in \mathbb{R} \\ \frac{\partial u}{\partial t}(x, 0) \; &= \; 0 \qquad x \in \mathbb{R}. \end{aligned}$$

4.7 Solve: Minimize $z = 2x_1 + 3x_2$, such that

$$\begin{aligned} x_1 + x_2 &\leq 4 \\ 3x_1 + x_2 &\geq 4 \\ x_1 + 5x_2 &\geq 4 \end{aligned}$$

and such that $0 \leq x_1 \leq 3$, and $0 \leq x_2 \leq 3$.

4.8 Consider the iterative scheme $x_{n+1} = Bx_n + c$ for $n \geq 0$, where B is a real $N \times N$ matrix and $c \in \mathbb{R}^N$. The scheme is said to be convergent if the sequence $\{x_n\}$ of iterates converges for *every* choice of initial vector x_0. Pick out the true statements.

a. The scheme is convergent if, and only if, the spectral radius of B is < 1.

b. The scheme is convergent if, and only if, for some matrix norm $\|.\|$, we have $\|B\| < 1$.

c. The scheme is convergent if, and only if, B has an eigenvalue λ such that $0 < \lambda < 1$.

4.9 Write down the Laplace transform $L[f](p)$ of the function $f(x) = \sin ax$, where $a > 0$.

4.10 What is the necessary and sufficient condition for the following problem to admit a solution?

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= g \quad \text{on } \partial\Omega \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega$, Δ is the Laplace operator, f and g are given smooth functions and $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of u.

Section 5: MISCELLANEOUS

5.1 Find the area of the polygon whose vertices are the n-th roots of unity in the complex plane, when $n \geq 3$.

5.2 Define $p_n(t) = \cos(n \cos^{-1} t)$ for $t \in [-1, 1]$. Express $p_4(t)$ as a polynomial in t.

5.3 What is the probability that a point (x, y), chosen at random in the rectangle $[-1, 1] \times [0, 1]$ is such that $y > x^2$?

5.4 An urn contains four white balls and two red balls. A ball is drawn at random and is replaced in the urn each time. What is the probability that after two successive draws, both balls drawn are white?

5.5 Let ABC be a triangle in the plane such that BC is perpendicular to AC. Let a, b, c be the lengths of BC, AC and AB respectively. Suppose that a, b, c are integers and have no common divisor other than 1. Which of the following statements are necessarily true?
a. Either a or b is an even integer.
b. The area of the triangle ABC is an even integer.
c. Either a or b is divisible by 3.

5.6 What are the last two digits in the usual decimal representation of 3^{400}?

5.7 Find the number of integers less than 3600 and prime to it.

5.8 Let n be a positive integer. Give an example of a sequence of n consecutive composite numbers.

5.9 For a point $P = (x, y)$ in the plane, write $f(P) = ax + by$, where a and b are given real numbers. Let $f(A) = f(B) = 10$. Let C be a point not on the line joining A and B and let C' be the reflection of C with respect to this line. If $f(C) = 15$, find $f(C')$.

5.10 Let V be a four dimenional vector space over the field $\mathbb{F}_3$ of three elements. Find the number of distinct one-dimensional subspaces of V.