NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 23, 2010

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions on the following page

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 11 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. **There will be no partial credit**.
- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, which is assumed to be endowed with its 'usual' topology. The symbol $\mathbb{Z}_n$ will denote the ring of integers modulo n. The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol I will denote the identity matrix of appropriate order. The space of continuous real valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual 'sup' norm.

Section 1: Algebra

1.1 Solve the equation

$$x^4 - 2x^3 + 4x^2 + 6x - 21 \;=\; 0$$

given that two of its roots are equal in magnitude but opposite in sign.

1.2 Let G be a group. A subgroup H of G is called *characteristic* if $\varphi(H) \subset H$ for all automorphisms φ of G. Pick out the true statement(s):
(a) Every characteristic subgroup is normal.
(b) Every normal subgroup is characteristic.
(c) If N is a normal subgroup of a group G, and M is a characteristic subgroup of N, then M is a normal subgroup of G.

1.3 Let G be a group and let H and K be subgroups of G. The *commutator subgroup* (H, K) is defined as the smallest subgroup containing all elements of the form $hkh^{-1}k^{-1}$, where $h \in H$ and $k \in K$. Pick out the true statement(s):
(a) If H and K are normal subgroups, then (H, K) is a normal subgroup.
(b) If H and K are characteristic subgroups, then (H, K) is a characteristic subgroup.
(c) (G, G) is normal in G and $G/(G, G)$ is abelian.

1.4 Write the following permutation as a product of disjoint cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 1 & 2 \end{pmatrix}.$$

1.5 Pick out the true statement(s):
(a) The set of all 2×2 matrices with rational entries (with the usual operations of matrix addition and matrix multiplication) is a ring which has no non-trivial ideals.
(b) Let $R = \mathcal{C}[0, 1]$ be considered as a ring with the usual operations of pointwise addition and pointwise multiplication. Let

$$\mathcal{I} \;=\; \{f : [0, 1] \to \mathbb{R} \mid f(1/2) = 0\}.$$

Then $\mathcal{I}$ is a maximal ideal.
(c) Let R be a commutative ring and let $\mathcal{P}$ be a prime ideal of R. Then $R/\mathcal{P}$ is an integral domain.

1.6 What is the degree of the following numbers over $\mathbb{Q}$?
(a) $\sqrt{2} + \sqrt{3}$
(b) $\sqrt{2}\sqrt{3}$

1.7 Let V be the real vector space of all polynomials of degree ≤ 3 with real coefficients. Define the linear transformation

$$T(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) \;=\; \alpha_0 + \alpha_1(x+1) + \alpha_2(x+1)^2 + \alpha_3(x+1)^3.$$

Write down the matrix of T with respect to the basis $\{1, x, x^2, x^3\}$ of V.

1.8 Let A be an $n \times n$ upper triangular matrix with complex entries. Pick out the true statement(s):
(a) If $A \neq 0$, and if $a_{ii} = 0$, for all $1 \leq i \leq n$, then $A^n = 0$.
(b) If $A \neq I$ and if $a_{ii} = 1$ for all $1 \leq i \leq n$, then A is not diagonalizable.
(c) If $A \neq 0$, then A is invertible.

1.9 Pick out the true statement(s):
(a) There exist $n \times n$ matrices A and B with real entries such that

$$(I - (AB - BA))^n = 0.$$

(b) If A is a symmetric and positive definite $n \times n$ matrix, then

$$(\mathrm{tr}(A))^n \;\geq\; n^n \det(A)$$

where 'tr' denotes the trace and 'det' denotes the determinant of a matrix.
(c) Let A be a 5×5 skew -symmetric matrix with real entries. Then A is singular.

1.10 Let A be a 5×5 matrix whose characteristic polynomial is given by

$$(\lambda - 2)^3(\lambda + 2)^2.$$

If A is diagonalizable, find α and β such that

$$A^{-1} \;=\; \alpha A + \beta I.$$

Section 2: Analysis

2.1 Let $\{a_n\}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r < 1.$$

Can we evaluate $\lim_{n\to\infty} a_n$? If 'yes', right down that limit.

2.2 Test the following series for convergence:
(a)

$$\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+\frac{5}{4}}}.$$

(b)

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right).$$

2.3 Consider the polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

with real coefficients. Pick out the case(s) which ensure that the polynomial $p(.)$ has a root in the interval $[0,1]$.
(a) $a_0 < 0$ and $a_0 + a_1 + \cdots + a_n > 0$.
(b)

$$a_0 + \frac{a_1}{2} + \cdots + \frac{a_n}{n+1} = 0.$$

(c)

$$\frac{a_0}{1.2} + \frac{a_1}{2.3} + \cdots + \frac{a_n}{(n+1)(n+2)} = 0.$$

2.4 Pick out the true statement(s):
(a) The function

$$f(x) = \frac{\sin(x^2)}{\sin^2 x}$$

is uniformly continuous on the interval $]0,1[$.
(b) A continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if it maps Cauchy sequences into Cauchy sequences.
(c) If a continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then it maps Cauchy sequences into Cauchy sequences.

2.5 Test the following for uniform convergence:
(a) The sequence of functions

$$\left\{ \frac{x^n}{1+x^n} \right\}$$

over the interval $[0, 2]$.
(b) The series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2+1}$$

over $\mathbb{R}$.
(c) The sequence of functions

$$\{n^2 x^2 e^{-nx}\}$$

over the interval $]0, \infty[$.

2.6 Evaluate:

$$\lim_{n\to\infty} \left\{ \left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right) \right\}^{\frac{1}{n}}.$$

2.7 Let $f : [-\pi, \pi] \to \mathbb{R}$ be continuous. Pick out the case(s) which imply that $f \equiv 0$.
(a)

$$\int_{-\pi}^{\pi} x^n f(x)\, dx \;=\; 0, \quad \text{for all} \quad n \geq 0.$$

(b)

$$\int_{-\pi}^{\pi} f(x) \cos nx\, dx \;=\; 0, \quad \text{for all} \quad n \geq 0.$$

(c)

$$\int_{-\pi}^{\pi} f(x) \sin nx\, dx \;=\; 0, \quad \text{for all} \quad n \geq 1.$$

2.8 Evaluate:

$$\int_{\Gamma} \frac{dz}{(z^2+4)^2}$$

where $\Gamma = \{z \in \mathbb{C} \mid |z-i| = 2\}$, described in the anticlockwise (*i.e.* positive) direction.

2.9 Find the residue at $z = 1$ of the function:

$$f(z) \;=\; \frac{5z-2}{z(z-1)}.$$

2.10 Let $f : \mathbb{C} \to \mathbb{C}$ be analytic. Which of the following conditions imply that f is a constant function?
(a) $\text{Re} f(z) > 0$ for all $z \in \mathbb{C}$.
(b) $|f(z)| \in \mathbb{Z}$ for all $z \in \mathbb{C}$.
(c) $f(z) = i$ when $z = (1 + \frac{k}{n}) + i$ for every positive integer k.

Section 3: Topology

3.1 Let S^1 denote the unit circle in the plane $\mathbb{R}^2$. Pick out the true statement(s):
(a) There exists $f : S^1 \to \mathbb{R}$ which is continuous and one-one.
(b) For every continuous function $f : S^1 \to \mathbb{R}$, there exist uncountably many pairs of distinct points x and y in S^1 such that $f(x) = f(y)$.
(c) There exists $f : S^1 \to \mathbb{R}$ which is continuous and one-one and onto.

3.2 Which of the following metric spaces are separable?
(a) $\mathcal{C}[0,1]$ with its usual 'sup-norm' topology.
(b) The space ℓ^∞ of all bounded real sequences with the metric

$$d(x,y) \;=\; \sup_n |x_n - y_n|,$$

where $x = (x_n)$ and $y = (y_n)$.
(c) The space ℓ^2 of all square summable real sequences with the metric

$$d(x,y) \;=\; \left(\sum_{n=1}^{\infty} |x_n - y_n|^2\right)^{\frac{1}{2}},$$

where $x = (x_n)$ and $y = (y_n)$.

3.3 Which of the following sets are nowhere dense?
(a) The Cantor set in $[0,1]$.
(b) The xy-plane in $\mathbb{R}^3$.
(c) Any countable set in $\mathbb{R}$.

3.4 Pick out the true statement(s).
(a) If $f :]-1,1[\to \mathbb{R}$ is bounded and continuous, it is uniformly continuous.
(b) If $f : S^1 \to \mathbb{R}$ is continuous, it is uniformly continuous.
(c) If (X,d) is a metric space and $A \subset X$, then the function $f(x) = d(x,A)$ defined by

$$d(x,A) \;=\; \inf_{y \in A} d(x,y)$$

is uniformly continuous.

3.5 Which of the following maps define a homeomorphism?
(a) $f : \mathbb{R} \to]0,\infty[$, where $f(x) = e^x$.
(b) $f : [0,1] \to S^1$, where $f(t) = (\cos 2\pi t, \sin 2\pi t)$.
(c) Any map $f : X \to Y$ which is continuous, one-one and onto, if X is compact and Y is Hausdorff.

3.6 Consider the set of all $n \times n$ matrices with real entries as the space $\mathbb{R}^{n^2}$. Which of the following sets are compact?
(a) The set of all orthogonal matrices.
(b) The set of all matrices with determinant equal to unity.
(c) The set of all invertible matrices.

3.7 In the set of all $n \times n$ matrices with real entries, considered as the space $\mathbb{R}^{n^2}$, which of the following sets are connected?
(a) The set of all orthogonal matrices.
(b) The set of all matrices with trace equal to unity.
(c) The set of all symmetric and positive definite matrices.

3.8 Let X be an arbitrary topological space. Pick out the true statement(s):
(a) If X is compact, then every sequence in X has a convergent subsequence.
(b) If every sequence in X has a convergent subsequence, then X is compact.
(c) X is compact if, and only if, every sequence in X has a convergent subsequence.

3.9 Which of the following metric spaces are complete?
(a) The space $\mathcal{C}^1[0,1]$ of continuously differentiable real-valued functions on $[0,1]$ with the metric

$$d(f,g) = \max_{t\in[0,1]} |f(t) - g(t)|.$$

(b) The space of all polynomials in a single variable with real coefficients, with the same metric as above.
(c) The space $\mathcal{C}[0,1]$ with the metric

$$d(f,g) = \int_0^1 |f(t) - g(t)| \, dt.$$

3.10 Classify the following alphabets into homeomorphism classes:

N, **B**, **H**, **M**

Section 4: Applied Mathematics

4.1 A body, falling under gravity, experiences a resisting force of air proportional to the square of the velocity of the body. Write down the differential equation governing the motion satisfied by the distance $y(t)$ travelled by the body in time t.

4.2 Reduce the following differential equation to a linear system of first order equations:

$$\frac{d^2x}{dt^2} + P(t)\frac{dx}{dt} + Q(t)x \;=\; 0.$$

4.3 The volume of the unit ball in $\mathbb{R}^N$ is given by

$$\omega_N \;=\; \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$$

where $\Gamma(.)$ denotes the usual gamma function. Write down the explicit value of ω_5.

4.4 Consider the differential equation

$$(1+x)y' \;=\; py$$

where p is a constant. Assume that the equation has a power series solution $y = \sum_{n=0}^{\infty} a_n x^n$. Write down the recurrence relation for the coefficients a_n.

4.5 In the above problem, if $y(0) = 1$, use the above series to find a closed form solution to the equation.

4.6 Classify the following partial differential operators as elliptic, parabolic or hyperbolic:
(a) $5u_{xx} + 6u_{xy} + 2u_{yy}$.
(b) $2u_{xx} + 6u_{xy} + 2u_{yy}$.

4.7 Let f and g be two smooth scalar valued functions. Compute

$$\mathrm{div}(\nabla f \times \nabla g).$$

4.8 Let S denote the sphere centred at the origin and of radius $a > 0$ in $\mathbb{R}^3$.Write down the coordinates of the unit outward normal to S at the point $(x, y, z) \in S$.

4.9 Use Gauss' divergence thoerem to evaluate

$$\int\int_S (x^4 + y^4 + z^4)\; dS$$

where S is the sphere mentioned in the preceding problem.

4.10 Consider the domain $[0,1] \times [0,T]$. Let $h > 0$ and $k > 0$. Let $x_n = nh$ and $t_m = mk$ for positive integers m and n. Let $u_n^m = u(x_n, t_m)$. Write down the partial differential equation for which the following discretization defines a numerical scheme:

$$\frac{u_n^{m+1} - u_n^m}{k} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{h^2}.$$

Section 5: Miscellaneous

5.1 Let n be a fixed positive integer and let C_k denote the usual binomial coefficient ${}^nC_k$, the number of ways of choosing k objects from n objects. Evaluate:

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}.$$

5.2 Find the number of ways $2n$ persons can be seated at 2 round tables, with n persons at each table.

5.3 Let a point (x, y) be chosen at random in the square $[0, 1] \times [0, 1]$. Find the probability that $y \geq x^2$.

5.4 Pick out the true statement(s):
(a) If n is an odd positive integer, then 8 divides $n^2 - 1$.
(b) If n and m are odd positive integers, then $n^2 + m^2$ is not a perfect square.
(c) For every positive integer n,

$$\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$$

is an integer.

5.5 Consider a circle of unit radius centered at O in the plane. Let AB be a chord which makes an angle θ with the tangent to the circle at A. Find the area of the triangle OAB.

5.6 Evaluate:

$$\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \cdots$$

5.7 Evaluate:

$$1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \cdots$$

5.8 Find the sum to n terms as well as the sum to infinity of the series:

$$\frac{1}{3} + \frac{1}{4}.\frac{1}{2!} + \frac{1}{5}.\frac{1}{3!} + \cdots$$

5.9 If a, b and c are all distinct real numbers, find the condition that the following determinant vanishes:

$$\begin{vmatrix} a & a^2 & 1 + a^3 \\ b & b^2 & 1 + b^3 \\ c & c^2 & 1 + c^3 \end{vmatrix}.$$

5.10 Assume that the line segment $[0, 2]$ in the x-axis of the plane acts as a mirror. A light ray from the point $(0, 1)$ gets reflected off this mirror and reaches the point $(2, 2)$. Find the point of incidence on the mirror.