**NATIONAL BOARD FOR HIGHER MATHEMATICS**

**Research Scholarships Screening Test**

**Saturday, January 22, 2011**

**Time Allowed: 150 Minutes**

**Maximum Marks: 40**

**Please read, carefully, the instructions on the following page**

## INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 11 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. **There will be no partial credit**.
- $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ - the integers, $\mathbb{Q}$ - the rationals, $\mathbb{R}$ - the reals and $\mathbb{C}$ - the field of complex numbers. $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, which is assumed to be endowed with its 'usual' topology. The symbol $]a,b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a,b]$ will stand for the corresponding closed interval; $[a,b[$ and $]a,b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol $I$ will denote the identity matrix of appropriate order. The space of continuous real valued functions on an interval $[a,b]$ is denoted by $\mathcal{C}[a,b]$ and is endowed with its usual 'sup' norm.

## Section 1: Algebra

**1.1** Solve:

$$x^4 - 3x^3 + 4x^2 - 3x + 1 \;=\; 0.$$

**1.2** Pick out the true statements:
a. Let $H$ and $K$ be subgroups of a group $G$. For $g \in G$, define the double coset

$$HgK \;=\; \{hgk \mid h \in H, k \in K\}.$$

Then, if $H$ is normal, we have $HgH = gH$ for all $g \in G$.
b. Let $GL(n;\mathbb{C})$ be the group of all $n \times n$ invertible matrices with complex entries. The set of all $n \times n$ invertible upper triangular matrices is a normal subgroup.
c. Let $\mathbb{M}(n;\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries (identified with $\mathbb{R}^{n^2}$ and endowed with its usual topology) and let $GL(n;\mathbb{R})$ denote the group of invertible matrices. Let $G$ be a subgroup of $GL(n;\mathbb{R})$. Define

$$H \;=\; \left\{ A \in G \mid \begin{array}{c} \text{there exists } \varphi : [0,1] \to G \text{ continuous,} \\ \text{such that } \varphi(0) = A, \varphi(1) = I \end{array} \right\}.$$

Then, $H$ is a normal subgroup of $G$.

**1.3** How many (non-isomorphic) groups of order 15 are there?

**1.4** Pick out the true statements:
a. Let $R$ be a commutative ring with identity. Let $M$ be an ideal such that every element of $R$ not in $M$ is a unit. Then $R/M$ is a field.
b. Let $R$ be as above and let $M$ be an ideal such that $R/M$ is an integral domain. Then $M$ is a prime ideal.
c. Let $R = \mathcal{C}[0,1]$ be the ring of real-valued continuous functions on $[0,1]$ with respect to pointwise addition and pointwise multiplication. Let

$$M \;=\; \{f \in R \mid f(0) = f(1) = 0\}.$$

Then $M$ is a maximal ideal.

**1.5** Write down all the possible values for the degree of an irreducible polynomial in $\mathbb{R}[x]$.

**1.6** Let $V$ be the real vector space of all polynomials in $\mathbb{R}[x]$ with degree less than, or equal to 4. Consider the linear transformation which maps $p \in V$ to its derivative $p'$. If the matrix of this transformation with respect to the basis $\{1, x, x^2, x^3, x^4\}$ is $A$, write down the matrix $A^3$.

**1.7** Let $\mathbb{T}(n;\mathbb{R}) \subset \mathbb{M}(n;\mathbb{R})$ denote the set of all matrices whose trace is zero. Write down a basis for $\mathbb{T}(2;\mathbb{R})$.

**1.8** What is the quotient space $\mathbb{M}(n;\mathbb{R})/\mathbb{T}(n;\mathbb{R})$ isomorphic to?

**1.9** Construct a $2 \times 2$ matrix $A(\neq I)$ with real entries such that $A^3 = I$.

**1.10** If $A \in \mathbb{M}(n;\mathbb{R})$, let ${}^tA$ denote its transpose. A matrix $S \in \mathbb{M}(n;\mathbb{R})$ is said to be *skew-symmetric* if ${}^tS = -S$. Pick out the true statements:
a. If $S \in \mathbb{M}(n;\mathbb{R})$ is skew-symmetric and non-singular, then $n$ is even.
b. Let

$$G = \{T \in GL(n;\mathbb{R}) \mid {}^tTST = S, \quad \text{for all skew-symmetric } S \in \mathbb{M}(n;\mathbb{R})\}.$$

Then $G$ is a subgroup of $GL(n;\mathbb{R})$.
c. Let $I_n$ and $O_n$ denote the $n \times n$ identity and null matrices respectively. let $S$ be the $2n \times 2n$ matrix given in block form by

$$\begin{bmatrix} O_n & I_n \\ -I_n & O_n \end{bmatrix}.$$

If $X$ is a $2n \times 2n$ matrix such that ${}^tXS + SX = 0$, then the trace of $X$ is zero.

## Section 2: Analysis

**2.1** Let $\{a_n\}$ be a sequence of positive terms. Pick out the cases which imply that $\sum a_n$ is convergent.

a.

$$\lim_{n\to\infty} n^{\frac{3}{2}} a_n = \frac{3}{2}.$$

b.

$$\sum n^2 a_n^2 < \infty.$$

c.

$$\frac{a_{n+1}}{a_n} < \left(\frac{n}{n+1}\right)^2, \text{ for all } n.$$

**2.2** Evaluate:

$$\lim_{n\to\infty} \left\{ \frac{1}{1+n^3} + \frac{4}{8+n^3} + \cdots + \frac{n^2}{n^3+n^3} \right\}.$$

**2.3** Find the points in $\mathbb{R}$ where the following function is differentiable:

$$f(x) = \begin{cases} \tan^{-1} x, & \text{if } |\mathrm{x}| \leq 1 \\ \frac{\pi}{4} sgn(x) + \frac{|x|-1}{2}, & \text{if } |\mathrm{x}| > 1, \end{cases}$$

where $sgn(x)$ equals $+1$ if $x > 0$, $-1$ if $x < 0$ and is equal to zero if $x = 0$ and $\tan^{-1}(x)$ takes its values in the range $]-\pi/2, \pi/2[$ for real numbers $x$.

**2.4** Pick out the true statements:

a. If $P$ is a polynomial in one variable with real coefficients which has all its roots real, then its derivative $P'$ has all its roots real as well.

b. The equation $\cos(\sin x) = x$ has exactly one solution in the interval $[0, \frac{\pi}{2}]$.

c. $\cos x > 1 - \frac{x^2}{2}$ for all $x > 0$.

**2.5** Let $f, f_n : [0,1] \to \mathbb{R}$ be continuous functions. Complete the following sentence such that both statements (a) and (b) below are true:

"Let $f_n \to f$ ......."

a.

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 f(x)\, dx.$$

b.

$$\lim_{n\to\infty} \lim_{x\to 0} f_n(x) = \lim_{x\to 0} \lim_{n\to\infty} f_n(x).$$

**2.6** Let $f : ]0,1[ \to \mathbb{R}$ be continuous. Pick out the statements which imply that $f$ is uniformly continuous.

a. $|f(x) - f(y)| \leq \sqrt{|x-y|}$, for all $x, y \in ]0,1[$.

b. $f(1/n) \to 1/2$ and $f(1/n^2) \to 1/4$.

c.

$$f(x) = x^{\frac{1}{2}} \sin \frac{1}{x^3}.$$

**2.7** Evaluate:

$$\int\int_{[0,1]\times[0,1]} \max\{x, y\}\ dxdy.$$

**2.8** Let $f : \mathbb{C} \to \mathbb{C}$ be analytic. Pick out the cases when $f$ is **not** necessarily a constant.
a. $\mathrm{Im}(\mathrm{f}'(\mathrm{z})) > 0$ for all $z \in \mathbb{C}$.
b. $f(n) = 3$ for all $n \in \mathbb{Z}$.
c. $f'(0) = 0$ and $|f'(z)| \leq 3$ for all $z \in \mathbb{C}$.

**2.9** Let $f : \mathbb{C} \to \mathbb{C}$ be analytic. Write $z = x + iy$ and $f = u + iv$, where $u$ and $v$ are real valued functions of $x$ and $y$. Pick out the true statements.
a.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

b.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

c.

$$f''(0) = \frac{1}{2\pi i}\int_{|z|=1} \frac{f(z)}{z^3}\ dz.$$

**2.10** Find the square roots of $1 + i\sqrt{3}$.

## Section 3: Topology

**3.1** Which of the following define a metric?
a. $d((x,y),(x',y')) = \min\{|x-x'|, |y-y'|\}$ on $\mathbb{R}^2$.
b. $d((x,y),(x',y')) = |x|+|y|+|x'|+|y'|$ on $\mathbb{R}^2$.
c. $D((x,y),(x',y')) = d(x,x') + d(y,y')$ on $X \times X$, where $(X,d)$ is a metric space.

**3.2** Let $(X,d)$ be a metric space and let $A \subset X$. For $x \in X$ define

$$d(x,A) \;=\; \inf\{d(x,y) \mid y \in A\}.$$

Pick out the true statements:
a. $x \mapsto d(x,A)$ is a uniformly continuous function.
b. If

$$\partial A \;=\; \{x \in X \mid d(x,A) = 0\} \cap \{x \in X \mid d(x, X\backslash A) = 0\},$$

then $\partial A$ is closed for any $A \subset X$.
c. Let $A$ and $B$ be subsets of $X$ and define

$$d(A,B) \;=\; \inf\{d(a,B) \mid a \in A\}.$$

Then $d(A,B) = d(B,A)$.

**3.3** Let $X$ be a topological space and for $A \subset X$, denote by $\overline{A}$ and $A^\circ$, the closure and interior of $A$ respectively. Pick out the true statements.
a. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
b. $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
c. Consider $\mathbb{R}$ as the $x$-axis in $\mathbb{R}^2$. Then $\mathbb{R}^\circ = \emptyset$.

**3.4** Pick out the true statements.
a. Let $\{X_i\}_{i \in \mathcal{I}}$ be topological spaces. Then, the product topology is the smallest topology on $X = \Pi_{i\in\mathcal{I}} X_i$ such that each of the canonical projections $p_i : X \to X_i$ is continuous.
b. Let $X$ be a topological space and $W \subset X$. Then, the induced subspace topology on $W$ is the smallest topology such that $id|_W : W \to X$, where $id$ is the identity map, is continuous.
c. Let $X = \mathbb{R}^n$ with the usual topology. This is the smallest topology such that all linear functionals on $X$ are continuous.

**3.5** Which of the following subsets are dense in the given spaces?
a. The set of trigonometric polynomials in the space of continuous functions on $[-\pi, \pi]$ which are $2\pi$-periodic (with the sup-norm topology).
b. The subset of $\mathcal{C}^\infty$ functions with compact support in $\mathbb{R}$ in the space of bounded real-valued continuous functions on $\mathbb{R}$ (with the sup-norm topology).
c. $GL(n; \mathbb{R})$ in $\mathbb{M}(n; \mathbb{R})$ (with its usual topology after identification with $\mathbb{R}^{n^2}$).

**3.6** Pick out the compact sets.
a. $\{(x, y) \mid x^2 - y^2 = 1\} \subset \mathbb{R}^2$.
b. $\{\mathrm{Tr(A)} \mid \mathrm{A} \in \mathbb{M}(\mathrm{n}; \mathbb{R}), \mathrm{A \text{ orthogonal}}\} \subset \mathbb{R}$, where Tr(A) denotes the trace of the matrix $A$.
c. The set of all matrices in $\mathbb{M}(n; \mathbb{R})$ all of whose eigenvalues satisfy the condition $|\lambda| \leq 2$.

**3.7** Pick out the connected sets.
a. $\{(x, y) \mid xy = 1\} \subset \mathbb{R}^2$.
b. The set of all upper triangular matrices in $\mathbb{M}(n; \mathbb{R})$.
c. The set of all invertible diagonal matrices in $\mathbb{M}(n; \mathbb{R})$.

**3.8** Pick out the true statements.
a. Let $f : \mathbb{Z} \to \mathbb{Z}^2$ be a bijection. There exists a continuous function from $\mathbb{R}$ to $\mathbb{R}^2$ which extends $f$.
b. Let $D$ denote the closed unit disc in $\mathbb{R}^2$. There exists a continuous mapping $f : D\backslash\{(0, 0)\} \to \{x \in \mathbb{R} \mid \ |x| \leq 1\}$ which is onto.
c. Let $D$ denote the closed unit disc in $\mathbb{R}^2$. There exists a continuous mapping $f : D\backslash\{(0, 0)\} \to \{x \in \mathbb{R} \mid \ |x| > 1\}$ which is onto.

**3.9** A Hausdorff topological space is said to be normal if given any two disjoint closed sets $A$ and $B$, there exist disjoint open sets $U$ and $V$ such that $A \subset U$ and $B \subset V$. Pick out the true statements.
a. Every metric space is normal.
b. If $X$ is a normal space with at least two distinct points, then there exist non-constant real-valued continuous functions on $X$.
c. If $X$ is normal and $Y \subset X$ is closed, then $Y$ is normal for the induced topology.

**3.10** Which of the following pairs of sets are homeomorphic?
a. $A = \{(x, y) \mid x^2 + y^2 - 2x + 4y - 5 = 0\}$ and
$B = \{(x, y) \ |5x^2 + 3y^2 = 1\}$.
b. $A = \{(x, y) \mid x^2 + y^2 - 2x + 4y - 5 = 0\}$ and
$B = \{(x, y) \ |5x^2 - 3y^2 = 1\}$.
c. $A = \{(x, y) \mid x^2 + y^2 - 2x + 4y - 5 \leq 0\}$ and
$B = \{(x, y) \ |5x^2 + 3y^2 \geq 1\}$.

## Section 4: Applied Mathematics

**4.1** Simpson's rule is used to approximate the integral $\int_0^1 f(x)\, dx$. If $f$ is a polynomial, what is the maximum possible degree it can have so that Simpson's rule gives the exact value of this integral?

**4.2** A right circular cylinder of fixed volume has maximum total surface area. What is the relationship between its height $h$ and radius $r$?

**4.3** In the equations governing the flow of an incompressible fluid of uniform density, if $\mathbf{u}$ is the velocity vector and $p$ is the pressure, write down the equation which expresses the law of conservation of mass.

**4.4** A particle of mass $M$ is attached to a fixed wall by a spring. The spring exerts no force when the particle is at its equilibrium position at $x = 0$ and exerts a restoring force proportional to the displacement when it is displaced to a distance $x$. In addition, there is a damping force due to the medium in which the displacement takes place, which is a force opposing the motion and is proportional to the velocity of the particle. If the particle is pulled to a position $x_0$ at time $t = 0$ and is released without any velocity, write down the initial value problem governing the motion of the particle.

**4.5** Solve the following linear programming problem:

$$\begin{array}{rcl} \max z & = & 5x + 7y \\ x - y & \leq & 1 \\ 2x + y & \geq & 2 \\ x + 2y & \leq & 4 \\ x, y & \geq & 0. \end{array}$$

**4.6** Write down the dual of the above problem.

**4.7** Find the general solution of the system:

$$\begin{array}{rcl} \frac{dx}{dt} & = & x + y \\ \frac{dy}{dt} & = & 4x - 2y. \end{array}$$

**4.8** Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbb{R}^3$. Express $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ as a linear combination of $\mathbf{b}$ and $\mathbf{c}$.

**4.9** Solve:

$$\frac{\partial^2 u}{\partial x^2} = 6xy;\ u(0, y) = y;\ \frac{\partial u}{\partial x}(1, y) = 0.$$

**4.10** Let $\Omega$ be a smooth plane domain of unit area. Let $u(x, y) = 3x^2 + y^2$. If $\frac{\partial u}{\partial n}$ denotes its outer normal derivative on $\partial\Omega$, the boundary of $\Omega$, compute

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}\, ds.$$

## Section 5: Miscellaneous

**5.1** Let $V$ be a real vector space of real-valued functions on a given set. Assume that constant functions are in $V$ and that if $f \in V$, then $f^2 \in V$ and that $|f| \in V$. Pick out the true statements.
a. If $f, g \in V$, then $fg \in V$.
b. If $f, g \in V$, then $\max\{f, g\} \in V$.
c. If $f \in V$ and $p$ is any polynomial in one variable, with real coefficients, then $p(f) \in V$.

**5.2** A fair coin is tossed 10 times, the tosses being independent of each other. Find the probability that the results of the third, fourth and fifth tosses are identical.

**5.3** Determine if the following collections are countable or uncountable.
a. The collection of all finite subsets of $\mathbb{N}$.
b. The collection of all infinite sequences of positive integers.
c. The collection of all roots of all polynomials in one variable, with integer coefficients.

**5.4** Find the maximum value of $x + 2y + 3z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

**5.5** Let $A_n$ be the $n \times n$ matrix whose $(i, j)$-th entry is given by

$$2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1}$$

where $\delta_{ij}$ equals 1 if $i = j$ and zero otherwise. Compute the determinant of $A_n$.

**5.6** How many real roots does the following equation have?

$$3^x + 4^x = 5^x$$

**5.7** Let $N > 1$ be a positive integer. Let $\phi(N)$ denote the number of positive integers less than $N$ and prime to it (unity being included in this count). Express the sum of all the integers less than $N$ and prime to it in terms of $\phi(N)$.

**5.8** Pick out the true statements.
a. The sum of $r$ consecutive positive integers is divisible by $r$.
b. The product of $r$ consecutive positive integers is divisible by $r!$.
c. For each positive integer $r$, there exist $r$ consecutive positive integers which are all composite.

**5.9** Let $n$ be a fixed positive integer and let $0 \leq k \leq n$. We denote by $C_k$, the number of ways of choosing $k$ objects from n distinct objects. Sum to $n$ terms:

$$3C_1 + 7C_2 + 11C_3 + \cdots$$

**5.10** Find the sum of the following infinite series:

$$\frac{1}{5} - \frac{1.4}{5.10} + \frac{1.4.7}{5.10.15} - \cdots$$