

NATIONAL BOARD FOR HIGHER MATHEMATICS

Research Scholarships Screening Test

Saturday, January 21, 2017

Time Allowed: 150 Minutes

Maximum Marks: 40

Please read, carefully, the instructions that follow.

INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 11 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Calculus & Differential Equations, and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of **all** the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded for questions involving more than one answer only if **all** the correct answers are given. **There will be no partial credit.**
- Calculators are **not allowed**.

Notation

- \mathbb{N} denotes the set of natural numbers $\{1, 2, 3, \dots\}$, \mathbb{Z} - the integers, \mathbb{Q} - the rationals, \mathbb{R} - the reals and \mathbb{C} - the field of complex numbers.
- Let $n \in \mathbb{N}, n \geq 2$. The symbol \mathbb{R}^n (respectively, \mathbb{C}^n) denotes the n -dimensional Euclidean space over \mathbb{R} (respectively, over \mathbb{C}), and is assumed to be endowed with its ‘usual’ topology. $\mathbb{M}_n(\mathbb{R})$ (respectively, $\mathbb{M}_n(\mathbb{C})$) will denote the set of all $n \times n$ matrices with entries from \mathbb{R} (respectively, \mathbb{C}) and is identified with \mathbb{R}^{n^2} (respectively, \mathbb{C}^{n^2}) when considered as a topological space.
- The symbol $\binom{n}{r}$ will denote the standard binomial coefficient giving the number of ways of choosing r objects from a collection of n objects, where $n \geq 1$ and $0 \leq r \leq n$ are integers.
- The symbol $]a, b[$ will stand for the open interval $\{x \in \mathbb{R} \mid a < x < b\}$ while $[a, b]$ will stand for the corresponding closed interval; $[a, b[$ and $]a, b]$ will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively.
- The space of continuous real valued functions on an interval $[a, b]$ is denoted by $\mathcal{C}[a, b]$ and is endowed with its usual ‘sup-norm’ metric.
- The space of continuous real valued functions on \mathbb{R} which have compact support will be denoted $\mathcal{C}_c(\mathbb{R})$ and will be equipped with the ‘sup-norm’ metric.
- Let $1 \leq p < \infty$ and let $]a, b[\subset \mathbb{R}$ be an open interval equipped with the Lebesgue measure. The symbol $L^p(]a, b[)$ will stand for the space of measurable functions such that

$$\int_a^b |f(t)|^p dt < \infty.$$

The space $L^\infty(]a, b[)$ will stand for the space of essentially bounded functions. These spaces are equipped with their usual norms.

- The derivative of a function f is denoted by f' and the second derivative by f'' .
- The symbol I will denote the identity matrix of appropriate order.
- The determinant of a square matrix A will be denoted by $\det(A)$ and its trace by $\text{tr}(A)$.
- $GL_n(\mathbb{R})$ (respectively, $GL_n(\mathbb{C})$) will denote the group of invertible $n \times n$ matrices with entries from \mathbb{R} (respectively, \mathbb{C}) with the group operation being matrix multiplication. The symbol $SL_n(\mathbb{R})$ will denote the subgroup of $GL_n(\mathbb{R})$, of matrices whose determinant is unity.
- The symbol S_n will denote the group of all permutations of n symbols $\{1, 2, \dots, n\}$, the group operation being composition.
- The symbol \mathbb{Z}_n will denote the additive group of integers modulo n .
- The symbol \mathbb{F}_p will denote the field consisting of p elements, where p is a prime.
- Unless specified otherwise, all logarithms are to the base e .

Section 1: Algebra

1.1 Let G be a group. Which of the following statements are true?

- Let H and K be subgroups of G of orders 3 and 5 respectively. Then $H \cap K = \{e\}$, where e is the identity element of G .
- If G is an abelian group of odd order, then $\varphi(x) = x^2$ is an automorphism of G .
- If G has exactly one element of order 2, then this element belongs to the centre of G .

1.2 Let $n \in \mathbb{N}, n \geq 2$. Which of the following statements are true?

- Any finite group G of order n is isomorphic to a subgroup of $GL_n(\mathbb{R})$.
- The group \mathbb{Z}_n is isomorphic to a subgroup of $GL_2(\mathbb{R})$.
- The group \mathbb{Z}_{12} is isomorphic to a subgroup of S_7 .

1.3 Which of the following statements are true?

- a. The matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

are conjugate in $GL_2(\mathbb{R})$.

- b. The matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

are conjugate in $SL_2(\mathbb{R})$.

- c. The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

are conjugate in $GL_2(\mathbb{R})$.

1.4 Let p be an odd prime. Find the number of non-zero squares in \mathbb{F}_p .

1.5 Find a generator of \mathbb{F}_7^\times , the multiplicative group of non-zero elements of \mathbb{F}_7 .

1.6 The characteristic polynomial of a matrix $A \in M_5(\mathbb{R})$ is given by $x^5 + \alpha x^4 + \beta x^3$, where α and β are non-zero real numbers. What are the possible values of the rank of A ?

1.7 Let $A \in M_3(\mathbb{R})$ be a symmetric matrix whose eigenvalues are 1, 1 and 3. Express A^{-1} in the form $\alpha I + \beta A$, where $\alpha, \beta \in \mathbb{R}$.

1.8 Let $A \in M_n(\mathbb{R}), n \geq 2$. Which of the following statements are true?

- If $A^{2n} = 0$, then $A^n = 0$.
- If $A^2 = I$, then $A = \pm I$.
- If $A^{2n} = I$, then $A^n = \pm I$.

1.9 Which of the following statements are true?

- a. There does not exist a non-diagonal matrix $A \in \mathbb{M}_2(\mathbb{R})$ such that $A^3 = I$.
- b. There exists a non-diagonal matrix $A \in \mathbb{M}_2(\mathbb{R})$ which is diagonalizable over \mathbb{R} and which is such that $A^3 = I$.
- c. There exists a non-diagonal matrix $A \in \mathbb{M}_2(\mathbb{R})$ such that $A^3 = I$ and such that $\text{tr}(A) = -1$.

1.10 Let $n \geq 2$ and let W be the subspace of $\mathbb{M}_n(\mathbb{R})$ consisting of all matrices whose trace is zero. If $A = (a_{ij})$ and $B = (b_{ij})$, for $1 \leq i, j \leq n$, are elements in $\mathbb{M}_n(\mathbb{R})$, define their inner-product by

$$(A, B) = \sum_{i,j=1}^n a_{ij}b_{ij}.$$

Identify the subspace W^\perp of elements orthogonal to the subspace W .

Section 2: Analysis

2.1 Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Let $\alpha = \liminf_{n \rightarrow \infty} x_n$. Which of the following statements are true?

- For every $\varepsilon > 0$, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \leq \alpha + \varepsilon$ for all $k \in \mathbb{N}$.
- For every $\varepsilon > 0$, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \leq \alpha - \varepsilon$ for all $k \in \mathbb{N}$.
- There exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightarrow \alpha$ as $k \rightarrow \infty$.

2.2 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ is divergent. Which of the following series are convergent?

a.

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + na_n}.$$

b.

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + n^2 a_n}.$$

c.

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n}.$$

2.3 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ is convergent. Which of the following series are convergent?

a.

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n}.$$

b.

$$\sum_{n=1}^{\infty} \frac{a_n^{\frac{1}{4}}}{n^{\frac{4}{5}}}.$$

c.

$$\sum_{n=1}^{\infty} na_n \sin \frac{1}{n}.$$

2.4 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a given function. It is said to be *lower semi-continuous* (respectively *upper semi-continuous*) if the set $f^{-1}(] - \infty, \alpha])$ (respectively, the set $f^{-1}([\alpha, \infty[))$ is closed for every $\alpha \in \mathbb{R}$. Let f and g be two real valued functions defined on \mathbb{R} . Which of the following statements are true?

- If f and g are continuous, then $\max\{f, g\}$ is continuous.
- If f and g are lower semi-continuous, then $\max\{f, g\}$ is lower semi-continuous.
- If f and g are upper semi-continuous, then $\max\{f, g\}$ is upper semi-continuous.

2.5 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Which of the following statements are true?

- If f is continuously differentiable, then f is uniformly continuous.
- If f has compact support, then f is uniformly continuous.
- If $\lim_{|x| \rightarrow \infty} |f(x)| = 0$, then f is uniformly continuous.

2.6 Let $f :]0, 2[\rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \in]0, 2[\cap \mathbb{Q}, \\ 2x - 1, & \text{if } x \in]0, 2[\setminus \mathbb{Q}. \end{cases}$$

Check for the points of differentiability of f and evaluate the derivative at those points.

2.7 Let $\{f_n\}_{n=1}^\infty$ be a sequence of continuous real valued functions defined on \mathbb{R} which converges pointwise to a continuous real valued function f . Which of the following statements are true?

a. If $0 \leq f_n \leq f$ for all $n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(t) dt = \int_{-\infty}^{\infty} f(t) dt.$$

b. If $|f_n(t)| \leq |\sin t|$ for all $t \in \mathbb{R}$ and for all $n \in \mathbb{N}$, then

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(t) dt = \int_{-\infty}^{\infty} f(t) dt.$$

c. If $|f_n(t)| \leq e^t$ for all $t \in \mathbb{R}$ and for all $n \in \mathbb{N}$, then for all $a, b \in \mathbb{R}, a < b$,

$$\lim_{n \rightarrow \infty} \int_a^b f_n(t) dt = \int_a^b f(t) dt.$$

2.8 Which of the following statements are true?

a. The following series is uniformly convergent over $[-1, 1]$:

$$\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}.$$

b.

$$\lim_{n \rightarrow \infty} \int_{\frac{\pi}{2}}^{\pi} \frac{\sin nx}{nx^5} dx = \pi.$$

c. Define, for $x \in \mathbb{R}$,

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx^2}{1+n^3}.$$

Then f is a continuously differentiable function.

2.9 Write down the Laurent series expansion of the function

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

in the annulus $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$.

2.10 Which of the following statements are true?

a. There exists a non-constant entire function which is bounded on the real and imaginary axes of \mathbb{C} .

b. The ring of analytic functions on the open unit disc of \mathbb{C} (with respect to the operations of pointwise addition and pointwise multiplication) is an integral domain.

c. There exists an entire function f such that $f(0) = 1$ and such that $|f(z)| \leq \frac{1}{|z|}$ for all $|z| \geq 5$.

Section 3: Topology

3.1 Let (X, d) be a metric space and let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be arbitrary Cauchy sequences in X . Which of the following statements are true?

- The sequence $\{d(x_n, y_n)\}$ converges as $n \rightarrow \infty$.
- The sequence $\{d(x_n, y_n)\}$ converges as $n \rightarrow \infty$ only if X is complete.
- No conclusion can be drawn about the convergence of $\{d(x_n, y_n)\}$.

3.2 Which of the following statements are true?

- Let X be a set equipped with two topologies τ_1 and τ_2 . Assume that any given sequence in X converges with respect to the topology τ_1 if, and only if, it also converges with respect to the topology τ_2 . Then $\tau_1 = \tau_2$.
- Let (X, τ_1) and (Y, τ_2) be two topological spaces and let $f : X \rightarrow Y$ be a given map. Then f is continuous if, and only if, given any sequence $\{x_n\}_{n=1}^\infty$ such that $x_n \rightarrow x$ in X , we have $f(x_n) \rightarrow f(x)$ in Y .
- Let (X, τ) be a compact topological space and let $\{x_n\}_{n=1}^\infty$ be a sequence in X . Then, it has a convergent subsequence.

3.3 Which of the following statements are true?

- Let $n \geq 2$. The subset of nilpotent matrices in $\mathbb{M}_n(\mathbb{C})$ is closed in $\mathbb{M}_n(\mathbb{C})$.
- Let $n \geq 2$. The set of all matrices in $\mathbb{M}_n(\mathbb{C})$ which represent orthogonal projections is closed in $\mathbb{M}_n(\mathbb{C})$.
- The set of all matrices in $\mathbb{M}_2(\mathbb{R})$ such that both of their eigenvalues are purely imaginary, is closed in $\mathbb{M}_2(\mathbb{R})$.

3.4 Which of the following sets are dense?

- The set of all numbers of the form $\frac{m}{2^n}$ where $0 \leq m \leq 2^n$ and $n \in \mathbb{N}$, in the space $[0, 1]$.
- The set of all polynomial functions in the space $L^1(]0, 1[)$.
- The linear span of the family $\{\sin nt\}_{n=1}^\infty$ in the space $L^2(]-\pi, \pi[)$.

3.5 Let $n \geq 2$. Which of the following subsets are nowhere dense in $\mathbb{M}_n(\mathbb{R})$?

- The set $GL_n(\mathbb{R})$.
- The set of all matrices whose trace is zero.
- The set of all singular matrices.

3.6 Which of the following topological spaces are separable?

- Any real Banach space which admits a Schauder basis $\{u_n\}_{n=1}^\infty$.
- The space $\mathcal{C}[0, 1]$.
- The space $L^p(]0, 1[)$, where $1 \leq p \leq \infty$.

3.7 Which of the following sets are connected?

- The set of all points in the plane with at least one coordinate irrational.
- An infinite set X with the topology τ given by

$$\tau = \{X, \emptyset\} \cup \{A \subset X \mid X \setminus A \text{ is a finite set}\}.$$

- The set

$$K = \{f \in \mathcal{C}[0, 1] \mid \int_0^{\frac{1}{2}} f(t) dt - \int_{\frac{1}{2}}^1 f(t) dt = 1\}.$$

3.8 Which of the following statements are true?

- There exists a continuous bijection $f : [0, 1] \rightarrow [0, 1] \times [0, 1]$.
- There exists a continuous map $f : S^1 \rightarrow \mathbb{R}$ which is injective, where S^1 stands for the unit circle in the plane.
- There exists a continuous map $f : [0, 1] \rightarrow SL_2(\mathbb{R})$ which is surjective.

3.9 Which of the following statements are true?

- Let $g \in \mathcal{C}[0, 1]$ be fixed. Then the set

$$A = \{f \in \mathcal{C}[0, 1] \mid \int_0^1 f(t)g(t) dt = 0\}$$

is closed in $\mathcal{C}[0, 1]$.

- Let $g \in \mathcal{C}_c(\mathbb{R})$, be fixed. Then the set

$$A = \{f \in \mathcal{C}_c(\mathbb{R}) \mid \int_{-\infty}^{\infty} f(t)g(t) dt = 0\}$$

is closed in $\mathcal{C}_c(\mathbb{R})$.

- Let $g \in L^2(\mathbb{R})$ be fixed. Then the set

$$A = \{f \in L^2(\mathbb{R}) \mid \int_{-\infty}^{\infty} f(t)g(t) dt = 0\}$$

is closed in $L^2(\mathbb{R})$.

3.10 Which of the following statements are true?

- Let X be a compact topological space and let \mathcal{F} be a family of real valued functions defined on X with the following properties:

- If $f, g \in \mathcal{F}$, then $fg \in \mathcal{F}$, where $(fg)(x) = f(x)g(x)$ for all $x \in X$.
 - For every $x \in X$, there exists an open neighbourhood $U(x)$ of x and a function $f \in \mathcal{F}$ such that the restriction of f to $U(x)$ is identically zero.
- Then the function which is identically zero on all of X belongs to \mathcal{F} .

- Let

$$X = \{f : [0, 1] \rightarrow [0, 1] \mid |f(t) - f(s)| \leq |t - s| \text{ for all } s, t \in [0, 1]\}.$$

Define

$$d(f, g) = \max_{t \in [0, 1]} |f(t) - g(t)|$$

for $f, g \in X$. Then (X, d) is a compact metric space.

- Let $\{f_i\}_{i \in I}$ be a collection of functions in $\mathcal{C}[0, 1]$ such that given any finite subfamily of functions, its members vanish at some common point (which depends on that subfamily). Then there exists $x_0 \in [0, 1]$ such that $f_i(x_0) = 0$ for all $i \in I$.

Section 4: Calculus & Differential Equations

4.1 Let $x > 1$. Define

$$F(x) = \int_{x^2}^{x^3} \tan(xy^2) dy.$$

Differentiate F with respect to x .

4.2 Evaluate:

$$\int_{-\infty}^{\infty} e^{-2x^2} dx.$$

4.3 Let $\mathbf{n}(x, y, z)$ denote the unit outer normal vector on the surface S of the cylinder $x^2 + y^2 \leq 4$, $0 \leq z \leq 3$. Compute

$$\int_S \mathbf{v} \cdot \mathbf{n} dS$$

where $\mathbf{v}(x, y, z) = xz\mathbf{i} + 2yz\mathbf{j} + 3xy\mathbf{k}$.

4.4 Evaluate the line integral $\int_C Pdx + Qdy$, where C is the circle centered at the origin and of radius $a > 0$ (described in the counter-clockwise sense) in the plane and

$$P(x, y) = \frac{-y}{x^2 + y^2}, \quad Q(x, y) = \frac{x}{x^2 + y^2}.$$

4.5 Let Ω be a bounded open subset of \mathbb{R}^3 and let $\partial\Omega$ denote its boundary. Given sufficiently smooth real valued functions u and v on $\bar{\Omega}$, let $\frac{\partial u}{\partial n}$ and $\frac{\partial v}{\partial n}$ denote the outer normal derivatives of u and v respectively on $\partial\Omega$. Fill in the blank in the following identity:

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u \right) dS = \int_{\Omega} (\dots\dots\dots) dx dy dz.$$

4.6 Find the maximum value of $x^2 + xy$ subject to the condition $x^2 + y^2 \leq 1$.

4.7 Interchange the order of integration:

$$\int_{-1}^2 \int_{-x}^{2-x^2} f(x, y) dy dx.$$

4.8 Find all the non-trivial solutions (λ, u) (*i.e.* $u \not\equiv 0$), of the boundary value problem:

$$-u''(x) = \lambda u(x), 0 < x < 1, \text{ and } u(0) = u'(1) = 0.$$

4.9 Consider the initial value problem: $u'(t) = Au(t)$, $t > 0$, and $u(0) = u_0$, where u_0 is a given vector in \mathbb{R}^2 and

$$A = \begin{bmatrix} 1 & -2 \\ 1 & a \end{bmatrix}.$$

Find the range of values of a such that $|u(t)| \rightarrow 0$ as $t \rightarrow \infty$.

4.10 Let $u(x, t)$ be the solution of the wave equation:

$$\left. \begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, t > 0, \\ u(x, 0) &= u_0(x), \quad x \in \mathbb{R}, \\ u_t(x, 0) &= 0, \quad x \in \mathbb{R}. \end{aligned} \right\}$$

Let $u_0(x)$ be the function defined by

$$u_0(x) = \begin{cases} 1, & \text{if } |x| < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Compute $u(x, 1)$ at all points $x \in \mathbb{R}$ where it is continuous.

Section 5: Miscellaneous

5.1 Let $x \in \mathbb{R}$ and let $n \in \mathbb{N}$. Evaluate:

$$\sum_{k=0}^n \binom{n}{k} \sin \left(x + \frac{k\pi}{2} \right).$$

5.2 Let $n \in \mathbb{N}, n \geq 2$. Let $x_1, \dots, x_n \in]0, \pi[$. Set $x = (x_1 + \dots + x_n)/n$. Which of the following statements are true?

a.

$$\prod_{k=1}^n \sin x_k \geq \sin^n x.$$

b.

$$\prod_{k=1}^n \sin x_k \leq \sin^n x.$$

c. Neither (a) nor (b) is necessarily true.

5.3 Which of the following sets are convex?

a.

$$\{(x, y) \in \mathbb{R}^2 \mid xy \geq 1, x \geq 0, y \geq 0\}.$$

b.

$$\{(x, y) \in \mathbb{R}^2 \mid |x|^{\frac{1}{3}} + |y|^{\frac{1}{3}} \leq 1\}.$$

c.

$$\{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\}.$$

5.4 Find the area of the circle got by intersecting the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x + y + z = 1$.

5.5 Let $n \in \mathbb{N}, n \geq 3$. Find the area of the polygon with one vertex at $z = 1$ and whose other vertices are situated at the roots of the polynomial

$$1 + z + z^2 + \dots + z^{n-1}$$

in the complex plane.

5.6 Find the maximum value of $3x + 2y$ subject to the conditions:

$$2x + 3y \geq 6, \quad y - x \leq 2, \quad 0 \leq x \leq 3, \quad y \geq 0$$

.

5.7 A committee of six members is formed from a group of 7 men and 4 women. What is the probability that the committee contains

a. exactly two women?

b. at least two women?

5.8 Find the sum of the infinite series:

$$\frac{1}{2.3.4} + \frac{1}{4.5.6} + \frac{1}{6.7.8} + \dots$$

5.9 Find the remainder when 8^{130} is divided by 13.

5.10 Let $a_i \in \mathbb{R}, 1 \leq i \leq 4$. Evaluate:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1 & a_2 & a_3 & a_4 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_1^3 & a_2^3 & a_3^3 & a_4^3 \end{vmatrix}.$$