



MODERN ALGEBRA

CLASS ASSIGNMENT

DECEMBER - 2014

PART - B

- The number of conjugacy classes in the permutation group S_6 is
1. 12 2. 11 3. 10 4. 6
- Find the degree of the field extension $\mathbb{Q}(\sqrt{2}, \sqrt[4]{2}, \sqrt[8]{2})$ over \mathbb{Q} .
1. 4 2. 8 3. 14 4. 32
- Let G be the Galois group of a field with nine elements over its subfield with three elements. Then the number of orbits for the action of G on the field with nine elements is
1. 3 2. 5 3. 6 4. 9
- The number of surjective maps from a set of 4 elements to a set of 3 elements is
1. 36 2. 64 3. 69 4. 81
- In the group of all invertible 4×4 matrices with entries in the field of 3 elements, any 3-Sylow subgroup has cardinality
1. 3 2. 81 3. 243 4. 729

PART - C

- Let G be a nonabelian group. Then, its order can be
1. 25 2. 55 3. 125 4. 35
- Let $\mathbb{R}[x]$ be the polynomial ring over \mathbb{R} in one variable. Let $I \subseteq \mathbb{R}[x]$ be an ideal. Then
 - I is a maximal ideal if and only if I is a non-zero prime ideal
 - I is a maximal ideal if and only if the quotient ring $\mathbb{R}[x]/I$ is isomorphic to \mathbb{R} .
 - I is a maximal ideal if and only if $I = (f(x))$, where $f(x)$ is a non-constant irreducible polynomial over \mathbb{R}
 - I is a maximal ideal if and only if there exists a non-constant polynomial $f(x) \in I$ of degree ≤ 2
- Let G be a group of order 45. Then
 - G has an element of order 9
 - G has a subgroup of order 9
 - G has a normal subgroup of order 9

4. G has a normal subgroup of order 5

- Which of the following is/are true?
 - Given any positive integer n , there exists a field extension of \mathbb{Q} of degree n .
 - Given a positive integer n , there exist fields F and K such that $F \subseteq K$ and K is Galois over F with $[K:F]=n$.
 - Let K be a Galois extension of \mathbb{Q} with $[K:\mathbb{Q}] = 4$. Then there is a field L such that $K \supseteq L \supseteq \mathbb{Q}$, $[L:\mathbb{Q}] = 2$ and L is a Galois extension of \mathbb{Q} .
 - There is an algebraic extension K of \mathbb{Q} such that $[K:\mathbb{Q}]$ is not finite.

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PART - B

- Up to isomorphism, the number of abelian groups of order 108 is:
1. 12 2. 9
3. 6 4. 5
- Let D be the set of tuples (w_1, \dots, w_{10}) , where $w_i \in \{1, 2, 3\}$, $1 \leq i \leq 10$ and $w_i + w_{i+1}$ is an even number for each i with $1 \leq i \leq 9$. Then the number of elements in D is.
1. $2^{11}+1$ 2. $2^{10}+1$
3. $3^{10}+1$ 4. $3^{11}+1$
- The number of subfields of a field of cardinality 2^{100} is
1. 2 2. 4
3. 9 4. 100
- Let R be the ring $\mathbb{Z}[x]/((x^2+x+1)(x^3+x+1))$ and I be the ideal generated by 2 in R . What is the cardinality of the ring R/I ?
1. 27 2. 32
3. 64 4. Infinite.

PART - C

- Which of the following polynomials are irreducible in the ring $\mathbb{Z}[x]$ of polynomials in one variable with integer coefficients?
 - $x^2 - 5$
 - $1+(x+1) + (x+1)^2 + (x+1)^3 + (x+1)^4$
 - $1+x+x^2+x^3+x^4$
 - $1+x+x^2+x^3$



15. Determine which of the following polynomials are irreducible over the indicated rings.

1. $x^5 - 3x^4 + 2x^3 - 5x + 8$ over \mathbb{R} .
2. $x^3 + 2x^2 + x + 1$ over \mathbb{Q} .
3. $x^3 + 3x^2 - 6x + 3$ over \mathbb{Z} .
4. $x^4 + x^2 + 1$ over $\mathbb{Z}/2\mathbb{Z}$.

16. Let $\sigma: \{1,2,3,4,5\} \rightarrow \{1,2,3,4,5\}$ be a permutation (one - to - one and onto function) such that $\sigma^{-1}(j) \leq \sigma(j) \forall j, 1 \leq j \leq 5$. Then which of the following are true?

1. $\sigma \circ \sigma(j) = j$ for all $j, 1 \leq j \leq 5$.
2. $\sigma^{-1}(j) = \sigma(j)$ for all $j, 1 \leq j \leq 5$.
3. The set $\{k: \sigma(k) \neq k\}$ has an even number of elements.
4. The set $\{k: \sigma(k) = k\}$ has an odd number of elements.

17. If x, y and z are elements of a group such that $xyz = 1$, then

1. $yzx = 1$
2. $yxz = 1$
3. $zxy = 1$
4. $zyx = 1$

18. Which of the following primes satisfy the congruence $a^{24} \equiv 6a + 2 \pmod{13}$?

1. 41
2. 47
3. 67
4. 83

19. Let $C([0,1])$ be the ring of all real valued continuous functions on $[0,1]$. Which of the following statements are true?

1. $C([0,1])$ is an integral domain.
2. The set of all functions vanishing at 0 is a maximal ideal.
3. The set of all functions vanishing at both 0 and 1 is a prime ideal.
4. If $f \in C([0,1])$ is such that $(f(x))^n = 0$ for all $x \in [0,1]$ for some $n > 1$, then $f(x) = 0$ for all $x \in [0,1]$.

20. Which of the following cannot be the class equation of a group of order 10?

1. $1 + 1 + 1 + 2 + 5 = 10$.
2. $1 + 2 + 3 + 4 = 10$.
3. $1 + 2 + 2 + 5 = 10$.
4. $1 + 1 + 2 + 2 + 2 + 2 = 10$.

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21. Which of the following is an irreducible factor of $x^{12} - 1$ over \mathbb{Q} ?

1. $x^8 + x^4 + 1$.
2. $x^4 + 1$.
3. $x^4 - x^2 + 1$.
4. $x^5 - x^4 + x^3 - x^2 + x - 1$.

22. Let R be a Euclidean domain such that R is not a field. Then the polynomial ring $R[X]$ is always

1. a Euclidean domain
2. a principal ideal domain, but not a Euclidean domain.
3. a unique factorization domain, but not a principal ideal domain.
4. not a unique factorization domain.

23. What is the total number of positive integer solutions to the equation

$$(x_1 + x_2 + x_3)(y_1 + y_2 + y_3 + y_4) = 15?$$

1. 1
2. 2
3. 3
4. 4

24. A group G is generated by the elements x, y with the relations $x^3 = y^2 = (xy)^2 = 1$. The order of G is

1. 4.
2. 6.
3. 8.
4. 12.

25. Let G be a simple group of order 60. Then

1. G has six Sylow-5 subgroups
2. G has four Sylow-3 subgroups.
3. G has a cyclic subgroup of order 6.
4. G has a unique element of order 2.

PART - C

26. Let $\omega = \cos \frac{2\pi}{10} + i \sin \frac{2\pi}{10}$.

Let $K = \mathbb{Q}(\omega^2)$ and let $L = \mathbb{Q}(\omega)$. Then

1. $[L : \mathbb{Q}] = 10$
2. $[L : K] = 2$
3. $[K : \mathbb{Q}] = 4$
4. $L = K$

27. Let a_n denote the number of those permutations σ on $\{1, 2, \dots, n\}$ such that σ is a product of exactly two disjoint cycles. Then:

1. $a_5 = 50$
2. $a_4 = 14$
3. $a_5 = 40$
4. $a_4 = 11$

28. Which of the following quotient rings are fields?

1. $F_3[X]/(X^2 + X + 1)$, where F_3 is the finite field with 3 elements.
2. $\mathbb{Z}[X]/(X - 3)$
3. $\mathbb{Q}[X]/(X^2 + X + 1)$
4. $F_2[X]/(X^2 + X + 1)$ where F_2 is the finite field with 2 elements.

29. Which of the following intervals contains an integer satisfying the following three congruences:

- $x \equiv 2 \pmod{5}$, $x \equiv 3 \pmod{7}$ and $x \equiv 4 \pmod{11}$.
1. [401, 600]
 2. [601, 800]
 3. [801, 1000]
 4. [1001, 1200]



30. Let A denote the quotient ring $\mathbb{Q}[X]/(X^3)$. Then
1. There are exactly three distinct proper ideals in A .
 2. There is only one prime ideal in A .
 3. A is an integral domain
 4. Let f, g be in $\mathbb{Q}[X]$ such that $\overline{f} \cdot \overline{g} = 0$ in A . Here \overline{f} and \overline{g} denote the image of f and g respectively in A . Then $f(0) \cdot g(0) = 0$.

31. For $n \geq 1$, let $(\mathbb{Z}/n\mathbb{Z})^*$ be the group of units of $(\mathbb{Z}/n\mathbb{Z})$. Which of the following groups are cyclic?
1. $(\mathbb{Z}/10\mathbb{Z})^*$
 2. $(\mathbb{Z}/2^3\mathbb{Z})^*$
 3. $(\mathbb{Z}/100\mathbb{Z})^*$
 4. $(\mathbb{Z}/163\mathbb{Z})^*$

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PART - B

32. Which of the following statements is FALSE?
There exists an integer x such that:
1. $x \equiv 23 \pmod{1000}$ and $z \equiv 45 \pmod{6789}$
 2. $x \equiv 23 \pmod{1000}$ and $z \equiv 54 \pmod{6789}$
 3. $x \equiv 32 \pmod{1000}$ and $z \equiv 54 \pmod{9876}$
 4. $x \equiv 32 \pmod{1000}$ and $z \equiv 44 \pmod{9876}$
33. Let $G = (\mathbb{Z}/25\mathbb{Z})^*$ be the group of units (i.e. the elements that have a multiplicative inverse) in the ring $(\mathbb{Z}/25\mathbb{Z})$. Which of the following is a generator of G ?
1. 3
 2. 4
 3. 5
 4. 6
34. Let $p \geq 5$ be a prime. Then
1. $F_p \times F_p$ has at least five subgroups of order p .
 2. Every subgroup of $F_p \times F_p$ is of the form $H_1 \times H_2$ where H_1, H_2 are subgroup of F_p .
 3. Every subgroup of $F_p \times F_p$ is an ideal of the ring $F_p \times F_p$
 4. The ring $F_p \times F_p$ is a field.
35. Let p be a prime number. How many distinct sub – rings (with unity) of cardinality p does the field F_{p^2} have?
1. 0
 2. 1
 3. p
 4. p^2

PART - C

36. Consider the symmetric group S_{20} and its subgroups A_{20} consisting of all even permutations. Let H be a 7-Sylow subgroup of A_{20} . Pick each correct statement from below.
1. $|H| = 49$.
 2. H must be cyclic.
 3. H is a normal subgroup of A_{20} .
 4. Any 7-Sylow subgroup of S_{20} is a subset of A_{20} .
37. Let R be a commutative ring with unity, such that $R[X]$ is a UFD. Denote the ideal (X) of $R[X]$ by I . Pick each correct statement from below.
1. I is prime.
 2. If I is maximal, then $R[X]$ is a PID.
 3. If $R[X]$ is a Euclidean domain, then I is maximal.
 4. If $R[X]$ is a PID, then it is a Euclidean domain.
38. Let G be a finite abelian group of order n . Pick each correct statement from below.
1. If d divides n , there exists a subgroup of G of order d .
 2. If d divides n , there exists an element of order d in G .
 3. If every proper subgroup of G is cyclic, then G is cyclic.
 4. If H is a subgroup of G , there exists a subgroup N of G such that $G/N \cong H$.
39. Let p be a prime, Pick each correct statement from below. Up to isomorphism.
1. There are exactly two abelian groups of order p^2 .
 2. There are exactly two groups of order p^2 .
 3. There are exactly two commutative rings of order p^2 .
 4. There is exactly one integral domain of order p^2 .
40. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree ≥ 2 . Pick each correct statement from below.
1. If $f(x)$ is irreducible in $\mathbb{Z}[x]$, then it is irreducible in $\mathbb{Q}[x]$.
 2. If $f(x)$ is irreducible in $\mathbb{Q}[x]$, then it is irreducible in $\mathbb{Z}[x]$.
 3. If $f(x)$ is irreducible in $\mathbb{Z}[x]$, then for all primes p the reduction $\overline{f(x)}$ of $f(x)$ modulo p is irreducible in $F_p[x]$.



4. If $f(x)$ is irreducible in $\mathbb{Z}[x]$, then it is irreducible in $\mathbb{R}[x]$.

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PART - B

41. $(n-1)! \equiv -1 \pmod{n}$. We can conclude that
1. $n = p^k$ where p is prime, $k > 1$.
 2. $n = pq$ where p and q are distinct primes.
 3. $n = pqr$ where p, q, r are distinct primes.
 4. $n = p$ where p is a prime.
42. Let S_n denote the permutation group on n symbols and A_n be the subgroup of even permutations. Which of the following is true?
1. There exists a finite group which is not a subgroup of S_n for any $n \geq 1$.
 2. Every finite group is a subgroup of A_n for some $n \geq 1$.
 3. Every finite group is a quotient of A_n for some $n \geq 1$.
 4. No finite abelian group is a quotient of S_n for $n > 3$.

PART - C

43. Consider the following subsets of the group of 2×2 non-singular matrices over \mathbb{R} :
- $$G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad = 1 \right\}$$
- $$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}.$$
- Which of the following statements are correct?
1. G forms a group under matrix multiplication.
 2. H is a normal subgroup of G .
 3. The quotient group G/H is well-defined and is Abelian.
 4. The quotient group G/H is well defined and is isomorphic to the group of 2×2 diagonal matrices (over \mathbb{R}) with determinant 1.
44. Let \mathbb{C} be the field of complex numbers and \mathbb{C}^* be the group of non zero complex numbers under multiplication. Then which of the following are true?
1. \mathbb{C}^* is cyclic.
 2. Every finite subgroup of \mathbb{C}^* is cyclic.
 3. \mathbb{C}^* has finitely many finite subgroups.
 4. Every proper subgroup \mathbb{C}^* is cyclic.

45. Let R be a finite non-zero commutative ring with unity. Then which of the following statements are necessarily true?

1. Any non-zero element of R is either a unit or a zero divisor.
2. There may exist a non-zero element of R which is neither a unit nor a zero divisor.
3. Every prime ideal of R is maximal.
4. If R has no zero divisors then order of any additive subgroup of R is a prime power.

46. Which of the following statements are true?

1. \mathbb{Z} is a principle ideal domain.
2. $\mathbb{Z}[x, y] / \langle y+1 \rangle$ is a unique factorization domain.
3. If R is a principle ideal domain and p is a non-zero prime ideal, then R/p has finitely many prime ideals.
4. If R is a principle ideal domain, then any subring of R containing 1 is again a principal ideal domain.

47. Let R be a commutative ring with unity and $R[x]$ be the polynomial ring in one variable. For a non zero $f = \sum_{n=0}^N a_n x^n$, define $\omega(f)$ to be the smallest n such that $a_n \neq 0$. Also $\omega(0) = +\infty$. Then which of the following statements is/are true?

1. $\omega(f + g) \geq \min(\omega(f), \omega(g))$.
2. $\omega(fg) \geq \omega(f) + \omega(g)$.
3. $\omega(f + g) = \min(\omega(f), \omega(g))$, if $\omega(f) \neq \omega(g)$.
4. $\omega(fg) = \omega(f) + \omega(g)$, if R is an integral domain.

48. Let \mathbb{F}_2 be the finite field of order 2. Then which of the following statements are true?

1. $\mathbb{F}_2[x]$ has only finitely many irreducible elements.
2. $\mathbb{F}_2[x]$ has exactly one irreducible polynomial of degree 2.
3. $\mathbb{F}_2[x] / \langle x^2 + 1 \rangle$ is a finite dimensional vector space over \mathbb{F}_2 .
4. Any irreducible polynomial in $\mathbb{F}_2[x]$ of degree 5 has distinct roots in any algebraic closure of \mathbb{F}_2 .



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PART - B

49. Consider the ideal $I = (x^2 + 1, y)$ in the polynomial ring $\mathbb{C}[x, y]$. Which of the following statements is true?
1. I is a maximal ideal
 2. I is a prime ideal but not a maximal ideal
 3. I is a maximal ideal but not a prime ideal
 4. I is neither a prime ideal nor a maximal ideal

PART - C

50. For an integer $n \geq 2$, let S_n be the permutation group on n letters and A_n the alternating group. Let \mathbb{C}^* be the group of non-zero complex numbers under multiplication. Which of the following are correct statements?
1. For every integer $n \geq 2$, there is a non trivial homomorphism $\chi: S_n \rightarrow \mathbb{C}^*$.
 2. For every integer $n \geq 2$, there is a unique nontrivial homomorphism $\chi: S_n \rightarrow \mathbb{C}^*$.
 3. For every integer $n \geq 3$, there is a nontrivial homomorphism $\chi: A_n \rightarrow \mathbb{C}^*$.
 4. For every integer $n \geq 5$, there is a nontrivial homomorphism $\chi: A_n \rightarrow \mathbb{C}^*$.
51. Let $R = \{f: \{1, 2, \dots, 10\} \rightarrow \mathbb{Z}_2\}$ be the set of all \mathbb{Z}_2 -valued functions on the set $\{1, 2, \dots, 10\}$ of the first ten positive integers. Then R is commutative ring with pointwise addition and pointwise multiplication of functions. Which of the following statements are correct?
1. R has a unique maximal ideal
 2. every prime ideal of R is also maximal
 3. Number of proper ideals of R is 511
 4. every element of R is idempotent
52. Which of the following rings are principal ideal domains (PID)?
1. $\mathbb{Q}[x]$
 2. $\mathbb{Z}[x]$
 3. $(\mathbb{Z}/6\mathbb{Z})[x]$
 4. $(\mathbb{Z}/7\mathbb{Z})[x]$
53. Let G be a group of order 125. Which of the following statements are necessarily true?
1. G has a non-trivial abelian subgroup
 2. The centre of G is a proper subgroup
 3. The centre of G has order 5
 4. There is a subgroup of order 25

54. Let R be a non-zero ring with identity such that $a^2 = a$ for all $a \in R$. Which of the following statements are true?
1. There is no such ring
 2. $2a = 0$ for all $a \in R$
 3. $3a = 0$ for all $a \in R$
 4. $\mathbb{Z}/2\mathbb{Z}$ is a subring of R
55. Which of the following polynomials are irreducible in $\mathbb{Z}[x]$?
1. $x^4 + 10x + 5$
 2. $x^3 - 2x + 1$
 3. $x^4 + x^2 + 1$
 4. $x^3 + x + 1$

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PART - B

56. Let $f: \mathbb{Z} \rightarrow (\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/6\mathbb{Z})$ be the function $f(n) = (n \bmod 4, n \bmod 6)$. Then
1. $(0 \bmod 4, 3 \bmod 6)$ is in the image of f
 2. $(a \bmod 4, b \bmod 6)$ is in the image of f , for all even integers a and b
 3. image of f has exactly 6 elements
 4. kernel of $f = 24\mathbb{Z}$
57. The group S_3 of permutations of $\{1, 2, 3\}$ acts on the three dimensional vector space over the finite field \mathbb{F}_3 of three elements, by permuting the vectors in basis $\{e_1, e_2, e_3\}$ by σ . $e_i = e_{\sigma(i)}$, for all $\sigma \in S_3$. The cardinality of the set of vectors fixed under the above action is
1. 0
 2. 3
 3. 9
 4. 27
58. Let R be a subring of \mathbb{Q} containing 1. Then which of the following is necessarily true?
1. R is a principal ideal domain (PID)
 2. R contains infinitely many prime ideals
 3. R contains a prime ideal which is not a maximal ideal
 4. for every maximal ideal m in R , the residue field R/m is finite

PART - C

59. Let G be a finite abelian group and $a, b \in G$ with $\text{order}(a) = m$, $\text{order}(b) = n$. Which of the following are necessarily true ?
1. $\text{order}(ab) = mn$
 2. $\text{order}(ab) = \text{lcm}(m, n)$
 3. there is an element of G whose order is $\text{lcm}(m, n)$
 4. $\text{order}(ab) = \text{gcd}(m, n)$



60. Which of the following rings are principal ideal domains (PIDs) ?

1. $\mathbb{Z}[X]/\langle X^2 + 1 \rangle$
2. $\mathbb{Z}[X]$
3. $\mathbb{C}[X, Y]$
4. $\mathbb{R}[X, Y]/\langle X^2 + 1, Y \rangle$

61. For any prime number p , let A_p be the set of integers $d \in \{1, 2, \dots, 999\}$ such that the power of p in the prime factorisation of d is odd. Then the cardinality of

1. A_3 is 250
2. A_5 is 160
3. A_7 is 124
4. A_{11} is 82

62. Let $z = e^{\frac{2\pi i}{7}}$ and let $\theta = z + z^2 + z^4$. Then

1. $\theta \in \mathbb{Q}$
2. $\theta \in \mathbb{Q}(\sqrt{D})$ for some $D > 0$
3. $\theta \in \mathbb{Q}(\sqrt{D})$ for some $D < 0$
4. $\theta \in i\mathbb{R}$

63. Let F be a finite field and let K/F be a field extension of degree 6. Then the Galois group of K/F is isomorphic to

1. the cyclic group of order 6
2. the permutation group on $\{1, 2, 3\}$
3. the permutation group on $\{1, 2, 3, 4, 5, 6\}$
4. the permutation group on $\{1\}$

JUNE - 2018

PART - B

64. Let S_7 denote the group of permutations of the set $\{1, 2, 3, 4, 5, 6, 7\}$. Which of the following is true?

1. There are no elements of order 6 in S_7
2. There are no elements of order 7 in S_7
3. There are no elements of order 8 in S_7
4. There are no elements of order 10 in S_7

65. The number of group homomorphisms from \mathbb{Z}_{10} to \mathbb{Z}_{20} is

1. zero
2. one
3. five
4. Ten

PART - C

66. Let $G = S_3$ be the permutation group of 3 symbols. Then

1. G is isomorphic to a subgroup of a cyclic group
2. there exists a cyclic group H such that G maps homomorphically onto H

3. G is a product of cyclic groups
4. there exists a nontrivial group homomorphism from G to the additive group $(\mathbb{Q}, +)$ of rational numbers

67. Let S be the set of polynomials $f(x)$ with integer coefficients satisfying $f(x) \equiv 1 \pmod{x-1}$; $f(x) \equiv 0 \pmod{x-3}$. Which of the following statements are true?

1. S is empty
2. S is a singleton
3. S is a finite non-empty set
4. S is countably infinite

68. Which of the following statements are true?

1. The multiplicative group of a finite field is always cyclic
2. The additive group of a finite field is always cyclic
3. There exists a finite field of any given order
4. There exists at most one finite field (upto isomorphism) of any given order

69. Which of the following statements are true?

1. A subring of an integral domain is an integral domain
2. A subring of a unique factorization domain (U.F.D.) is a U.F.D.
3. A subring of a principal ideal domain (P.I.D.) is a P.I.D.
4. A subring of an Euclidean domain is an Euclidean domain

70. Let G be a group with $|G| = 96$. Suppose H and K are subgroups of G with $|H| = 12$ and $|K| = 16$. Then

1. $H \cap K = \{e\}$
2. $H \cap K \neq \{e\}$
3. $H \cap K$ is Abelian
4. $H \cap K$ is not Abelian

December - 2018

PART - B

71. The number of group homomorphisms from the alternating group A_5 to the symmetric group S_4 is:

1. 1
2. 12
3. 20
4. 6

72. Let $p \geq 23$ be a prime number such that the decimal expansion (base 10) of $\frac{1}{p}$ is periodic

with period $p-1$ (that is, $\frac{1}{p} = 0.\overline{a_1 a_2 \dots a_{p-1}}$)



with $a_i \in \{0,1,\dots,9\}$ for all i and for any $m, 1 \leq m < p-1$, $\frac{1}{p} \neq \overline{0.a_1a_2\dots a_m}$. Let $(\mathbb{Z}/p\mathbb{Z})^*$ denote the multiplicative group of integers modulo p . Then which of the following is correct ?

1. The order of $10 \in (\mathbb{Z}/p\mathbb{Z})^*$ is a proper divisor of $(p-1)$
2. The order of $10 \in (\mathbb{Z}/p\mathbb{Z})^*$ is $\frac{(p-1)}{2}$
3. The element $10 \in (\mathbb{Z}/p\mathbb{Z})^*$ is a generator of the group $(\mathbb{Z}/p\mathbb{Z})^*$
4. The group $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic but not generated by the element 10.

73. Given integers a and b , let $N_{a,b}$ denote the number of positive integers $k < 100$ such that $k \equiv a \pmod{9}$ and $k \equiv b \pmod{11}$. Then which of the following statements is correct?
1. $N_{a,b} = 1$ for all integers a and b
 2. There exist integers a and b satisfying $N_{a,b} > 1$.
 3. There exist integers a and b satisfying $N_{a,b} = 0$
 4. There exists integers a and b satisfying $N_{a,b} = 0$ and there exists integers c and d satisfying $N_{c,d} > 1$

PART - C

74. For any group G , let $\text{Aut}(G)$ denote the group of automorphisms of G . Which of the following are true?
1. If G is finite, then $\text{Aut}(G)$ is finite
 2. If G is cyclic, then $\text{Aut}(G)$ is cyclic
 3. If G is infinite, then $\text{Aut}(G)$ is infinite
 4. If $\text{Aut}(G)$ is isomorphic to $\text{Aut}(H)$, where G and H are two groups, then G is isomorphic to H
75. Let G be a group with the following property: Given any positive integers m, n and r there exist elements g and h in G such that $\text{order}(g) = m$, $\text{order}(h) = n$ and $\text{order}(gh) = r$. Then which of the following are necessarily true?
1. G has to be an infinite group
 2. G cannot be a cyclic group
 3. G has infinitely many cyclic subgroups
 4. G has to be a non-abelian group
76. Let R be the ring $\mathbb{C}[x]/(x^2 + 1)$. Pick the correct statements from below:
1. $\dim_{\mathbb{C}} R = 3$

2. R has exactly two prime ideals
3. R is a UFD
4. (x) is a maximal ideal of R

77. Let $f(x) = x^7 - 105x + 12$. Then which of the following are correct?
1. $f(x)$ is reducible over \mathbb{Q}
 2. There exists an integer m such that $f(m) = 105$
 3. There exists an integer m such that $f(m) = 2$
 4. $f(m)$ is not a prime number for any integer m

78. Let $\alpha = \sqrt[5]{2} \in \mathbb{R}$ and $\xi = \exp\left(\frac{2\pi i}{5}\right)$. Let $K = \mathbb{Q}(\alpha\xi)$. Pick the correct statements from below:
1. There exists a field automorphism σ of \mathbb{C} such that $\sigma(K) = K$ and $\sigma \neq \text{id}$
 2. There exists a field automorphism σ of \mathbb{C} such that $\sigma(K) \neq K$
 3. There exists a finite extension E of \mathbb{Q} such that $K \subseteq E$ and $\sigma(K) \subseteq E$ for every field automorphism σ of E
 4. For all field automorphisms σ of K , $\sigma(\alpha\xi) = \alpha\xi$

JUNE - 2019

PART - B

79. For any integer $n \geq 1$, let $d(n)$ = number of positive divisors of n
 $v(n)$ = number of distinct prime divisors of n
 $w(n)$ = number of prime divisors of n counted with multiplicity
 [for example: If p is prime, then $d(p) = 2$, $v(p) = v(p^2) = 1$, $w(p^2) = 2$]
1. If $n \geq 1000$ and $w(n) \geq 2$, then $d(n) > \log n$
 2. there exists n such that $d(n) > 3\sqrt{n}$
 3. for every n , $2^{v(n)} \leq d(n) \leq 2^{w(n)}$
 4. if $w(n) = w(m)$, then $d(n) = d(m)$
80. Consider the set of matrices
- $$G = \left\{ \begin{pmatrix} s & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z}, s \in \{-1, +1\} \right\}$$
- Then which of the following is true?
1. G forms a group under addition



2. G forms an abelian group under multiplication
3. Every element in G is diagonalizable over \mathbb{C}
4. G is finitely generated group under multiplication

81. Let R be a commutative ring with unity. Which of the following is true?
1. If R has finitely many prime ideals, then R is a field.
 2. If R has finitely many ideals, then R is finite
 3. If R is a P.I.D., then every subring of R with unity is a P.I.D.
 4. If R is an integral domain which has finitely many ideals, then R is a field.

PART - C

82. Let $a \in \mathbb{Z}$ be such that $a = b^2 + c^2$, where $b, c \in \mathbb{Z} \setminus \{0\}$. Then a cannot be written as
1. pd^2 , where $d \in \mathbb{Z}$ and p is a prime with $p \equiv 1 \pmod{4}$
 2. pd^2 , where $d \in \mathbb{Z}$ and p is a prime with $p \equiv 3 \pmod{4}$
 3. pqd^2 , where $d \in \mathbb{Z}$ and p, q are primes with $p \equiv 1 \pmod{4}, q \equiv 3 \pmod{4}$
 4. pqd^2 , where $d \in \mathbb{Z}$ and p, q are distinct primes with $p, q \equiv 3 \pmod{4}$

83. For any prime p , consider the group $G = GL_2(\mathbb{Z}/p\mathbb{Z})$. Then which of the following are true?
1. G has an element of order p
 2. G has exactly one element of order p
 3. G has no p -Sylow subgroups
 4. Every element of order p is conjugate

to a matrix $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where $a \in (\mathbb{Z}/p\mathbb{Z})^*$

84. Let $\mathbb{Z}[X]$ be the ring of polynomials over integers. Then the additive group $\mathbb{Z}[X]$ is
1. isomorphic to the multiplicative group \mathbb{Q}^+ of positive rational numbers
 2. isomorphic to the group of rational numbers \mathbb{Q} under addition
 3. countable
 4. uncountable

85. Let $X = (0, 1)$ be the open unit interval and $C(X, \mathbb{R})$ be the ring of continuous functions

from X to \mathbb{R} . For any $x \in (0, 1)$, let $I(x) = \{f \in C(X, \mathbb{R}) \mid f(x) = 0\}$. Then which of the following are true?

1. $I(x)$ is a prime ideal
2. $I(x)$ is a maximal ideal
3. Every maximal ideal of $C(X, \mathbb{R})$ is equal to $I(x)$ for some $x \in X$
4. $C(X, \mathbb{R})$ is an integral domain

86. Let $n \in \mathbb{Z}$. Then which of the following are correct?

1. $X^3 + nX + 1$ is irreducible over \mathbb{Z} for every n
2. $X^3 + nX + 1$ is reducible over \mathbb{Z} if $n \in \{0, -2\}$
3. $X^3 + nX + 1$ is irreducible over \mathbb{Z} if $n \notin \{0, -2\}$
4. $X^3 + nX + 1$ is reducible over \mathbb{Z} for infinitely many n

87. Let \mathbb{F}_{27} denote the finite field of size 27. For each $\alpha \in \mathbb{F}_{27}$, we define

$A_\alpha = \{1, 1 + \alpha, 1 + \alpha + \alpha^2, 1 + \alpha + \alpha^2 + \alpha^3, \dots\}$.

Then which of the following are true?

1. the number of $\alpha \in \mathbb{F}_{27}$ such that $|A_\alpha| = 26$ equals 12
2. $0 \in A_\alpha$ if and only if $\alpha \neq 0$
3. $|A_1| = 27$
4. $\bigcap_{\alpha \in \mathbb{F}_{27}} A_\alpha$ is a singleton set

DECEMBER 2019

PART - B

88. Let G be a group of order p^n , p a prime number and $n > 1$. Then which of the following is true?

- (1) Centre of G has at least two elements
- (2) G is always an Abelian group
- (3) G has exactly two normal subgroups (i.e., G is a simple group)
- (4) If H is any other group of order p^n , then G is isomorphic to H

89. Let S_5 be the symmetric group on five symbols. Then which of the following statements is false?

- (1) S_5 contains a cyclic subgroup of order 6
- (2) S_5 contains a non-Abelian subgroup of order 8



- (3) S_5 does not contain a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
(4) S_5 does not contain a subgroup of order 7

90. A permutation σ of $[n] = \{1, 2, \dots, n\}$ is called irreducible, if the restriction $\sigma|_{[k]}$ is not a permutation of $[k]$ for any $1 \leq k < n$. Let a_n be the number of irreducible permutations of $[n]$. Then $a_1 = 1$, $a_2 = 1$ and $a_3 = 3$. The value of a_4 is
(1) 12 (2) 13
(3) 14 (4) 15

PART – C

91. Let I be an ideal of \mathbb{Z} . Then which of the following statements are true?
(1) I is a principal ideal
(2) I is a prime ideal of \mathbb{Z}
(3) If I is a prime ideal of \mathbb{Z} , then I is a maximal ideal in \mathbb{Z}
(4) If I is a maximal ideal in \mathbb{Z} , then I is a prime ideal of \mathbb{Z}
92. Let $f(x) \in \mathbb{Z}[x]$ be a monic polynomial of degree n . Then which of the following are true?
(1) If $f(x)$ is irreducible in $\mathbb{Z}[x]$, then it is irreducible in $\mathbb{Q}[x]$
(2) If $f(x)$ is irreducible in $\mathbb{Q}[x]$, then it is irreducible in $\mathbb{Z}[x]$
(3) If $f(x)$ is reducible in $\mathbb{Z}[x]$, then it has a real root
(4) If $f(x)$ has a real root, then it is reducible in $\mathbb{Z}[x]$
93. Let $F[X]$ be the polynomial ring in one variable over a field F . Then which of the following statements are true?
(1) $F[X]$ is a UFD
(2) $F[X]$ is a PID
(3) $F[X]$ is a Euclidean domain
(4) $F[X]$ is a PID but is not an Euclidean domain
94. Let $C[0, 1]$ be the ring of all real valued continuous function on $[0, 1]$
Let

$$A = \left\{ f \in C[0,1] : f\left(\frac{1}{4}\right) = f\left(\frac{3}{4}\right) = 0 \right\}. \text{ The}$$

n which of the following statements are true?

- (1) A is an ideal in $C[0, 1]$ but is not a prime ideal in $C[0, 1]$
(2) A is a prime ideal in $C[0, 1]$
(3) A is a maximal ideal in $C[0, 1]$
(4) A is a prime ideal in $C[0, 1]$, but is not a maximal ideal in $C[0, 1]$
95. For a given integer k , which of the follow statements are false?
(1) If $k \pmod{72}$ is a unit in \mathbb{Z}_{72} , then $k \pmod{9}$ is a unit in \mathbb{Z}_9
(2) If $k \pmod{72}$ is a unit in \mathbb{Z}_{72} , then $k \pmod{8}$ is a unit in \mathbb{Z}_8
(3) If $k \pmod{8}$ is a unit in \mathbb{Z}_8 , then $k \pmod{72}$ is a unit in \mathbb{Z}_{72}
(4) If $k \pmod{9}$ is a unit in \mathbb{Z}_9 , then $k \pmod{72}$ is a unit in \mathbb{Z}_{72}

96. Let F be a field. Then which of the following statements are true?
(1) All extensions of degree 2 of F are isomorphic as fields
(2) All finite extensions of F of same degree are isomorphic as fields if $\text{Char}(F) > 0$
(3) All finite extensions of F of same degree are isomorphic as fields if F is finite
(4) All finite normal extensions of F are isomorphic as fields if $\text{Char}(F) = 0$

JUNE 2020

PART – B

97. Which of the following statements is true?
(1) Every even integer $n \geq 16$ divides $(n-1)! + 3$
(2) Every odd integer $n \geq 16$ divides $(n-1)!$
(3) Every even integer $n \geq 16$ divides $(n-1)!$
(4) For every integer $n \geq 16$, n^2 divides $n! + 1$
98. Let X be a non-empty set and $P(X)$ be the set of all subsets of X . On $P(X)$, defined two operations $*$ and Δ as follows: for $A, B \in P(X)$, $A * B = A \cap B$; $A \Delta B = (A \cup B) \setminus (A \cap B)$.
Which of the following statements is true?
(1) $P(X)$ is a group under $*$ as well as under Δ



- (2) $P(X)$ is a group under $*$, but not under Δ
- (3) $P(X)$ is a group under Δ , but not under $*$
- (4) $P(X)$ is neither a group under $*$ nor under Δ

99. Let $\varphi(n)$ be the cardinality of the set $\{a \mid 1 \leq a \leq n, (a, n) = 1\}$ where (a, n) denotes the gcd of a and n . Which of the following is NOT true?

- (1) There exist infinitely many n such that $\varphi(n) > \varphi(n + 1)$.
- (2) There exist infinitely many n such that $\varphi(n) < \varphi(n + 1)$
- (3) There exists $N \in \mathbb{N}$ such that $N > 2$ and for all $n > N$, $\varphi(N) < \varphi(n)$
- (4) The set $\left\{ \frac{\varphi(n)}{n} : n \in \mathbb{N} \right\}$ has finitely many limit points

PART - C

100. Which of the following statements are true?

- (1) \mathbb{Q} has countably many subgroups
- (2) \mathbb{Q} has uncountably many subsets
- (3) Every finitely generated subgroup of \mathbb{Q} is cyclic
- (4) \mathbb{Q} is isomorphic to $\mathbb{Q} \times \mathbb{Q}$ as groups

101. Let $SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}$
and for any prime p , let

$$\Gamma(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \right. \\ \left. \begin{matrix} a \equiv 1 \pmod{p}, d \equiv 1 \pmod{p} \\ c \equiv 0 \pmod{p}, b \equiv 0 \pmod{p} \end{matrix} \right\}$$

Which of the following are true?

- (1) $\Gamma(p)$ is a subgroup of $SL_2(\mathbb{Z})$
- (2) $\Gamma(p)$ is not a normal subgroup of $SL_2(\mathbb{Z})$
- (3) $\Gamma(p)$ has atleast two elements
- (4) $\Gamma(p)$ is uncountable

102. Let G be a finite group. Which of the following are true?

- (1) If $g \in G$ has order m and if $n \geq 1$ divides m , then G has a subgroup of order n .
- (2) If for any two subgroups A and B of G , either $A \subset B$ or $B \subset A$, then G is cyclic.
- (3) If G is cyclic, then for any two subgroups A and B of G , either $A \subset B$ or $B \subset A$
- (4) If for every positive integer m dividing $|G|$, G has a subgroup of order m , then G is abelian

103. Let R, S be commutative rings with unity, $f : R \rightarrow S$ be a surjective ring homomorphism,

$Q \subseteq S$ be a non-zero prime ideal. Which of the following statements are true?

- (1) $f^{-1}(Q)$ is a non-zero prime ideal in R
- (2) $f^{-1}(Q)$ is a maximal ideal in R if R is a PID
- (3) $f^{-1}(Q)$ is a maximal ideal in R if R is a finite commutative ring with unity
- (4) $f^{-1}(Q)$ is a maximal ideal in R if $x^5 = x$ for all $x \in R$

104. Consider the polynomial $f(x) = x^2 + 3x - 1$. Which of the following statements are true?

- (1) f is irreducible over $\mathbb{Z}[\sqrt{13}]$
- (2) f is irreducible over \mathbb{Q}
- (3) f is reducible over $\mathbb{Q}[\sqrt{13}]$
- (4) $\mathbb{Z}[\sqrt{13}]$ is a unique factorization domain

105. Let p be an odd prime such that $p \equiv 2 \pmod{3}$. Let \mathbb{F}_p be the field with p elements. Consider the subset E of $\mathbb{F}_p \times \mathbb{F}_p$ given by $E = \{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p : y^2 = x^3 + 1\}$. Which of the following are true?

- (1) E has atleast two elements
- (2) E has atmost $2p$ elements
- (3) E can have p^2 elements
- (4) E has atleast $2p$ elements

JUNE 2021

PART - B

106. Let $S = \{n : 1 \leq n \leq 999; 3|n \text{ or } 37|n\}$. How many integers are there in the set $S^c = \{n : 1 \leq n \leq 999; n \notin S\}$?

- (1) 639
- (2) 648
- (3) 666
- (4) 990



107. How many generators does a cyclic group of order 36 have?
- (1) 6 (2) 12
(3) 18 (4) 24

108. Which of the following statements is necessarily true for a commutative ring R with unity?
- (1) R may have no maximal ideals
(2) R can have exactly two maximal ideals
(3) R can have one or more maximal ideals but no prime ideals
(4) R has at least two prime ideals

PART – C

109. A positive integer n co-prime to 17, is called a primitive root modulo 17 if n^{k-1} is not divisible by 17 for all k with $1 \leq k < 16$. Let a, b be distinct positive integers between 1 and 16. Which of the following statements are true?

- (1) 2 is a primitive root modulo 17
(2) If a is a primitive root modulo 17, then a^2 is not necessarily a primitive Root modulo 17
(3) If a, b are primitive roots modulo 17, then ab is a primitive root modulo 17
(4) Product of primitive roots modulo 17 between 1 and 16 is congruent to 1 modulo 17

110. For a positive integer n , let $\Omega(n)$ denote the number of prime factors of n , counted with multiplicity. For instance, $\Omega(3) = 1$, $\Omega(6) = \Omega(9) = 2$. Let $p > 3$ be a prime number and let $N = p(p + 2)(p + 4)$. Which of the following statements are true?

- (1) $\Omega(N) \geq 3$
(2) There exist primes $p > 3$ such that $\Omega(N) = 3$
(3) p can never be the smallest prime divisor of N
(4) p can be the smallest prime divisor of N

111. Let G be a group of order 24. Which of the following statements are necessarily true?

- (1) G has a normal subgroup of order 3
(2) G is not a simple group
(3) There exists an injective group homomorphism from G to S_8
(4) G has a subgroup of index 4

112. Which of the following statements are true?

- (1) All finite field extensions of \mathbb{Q} are Galois
(2) There exists a Galois extension of \mathbb{Q} of degree 3
(3) All finite field extensions of \mathbb{F}_2 are Galois
(4) There exists a field extension of \mathbb{Q} of degree 2 which is not Galois

113. Let $f = a_0 + a_1X + \dots + a_nX^n$ be a polynomial with $a_i \in \mathbb{Z}$ for $0 \leq i \leq n$. Let p be a prime such that $p|a_i$ for all $1 < i \leq n$ and p^2 does not divide a_n . Which of the following statements are true?

- (1) f is always irreducible
(2) f is always reducible
(3) f can sometimes be irreducible and can sometimes be reducible
(4) f can have degree 1

JUNE 2022

PART – B

114. Let R be a ring and N be the set of nilpotent elements i.e. $N = \{x \in R \mid x^n = 0 \text{ for some } n \in \mathbb{N}\}$. Which of the following is true?

- (1) N is an ideal in R
(2) N is never an ideal in R
(3) If R is non-commutative, N is not an ideal
(4) If R is commutative, N is an ideal

115. Let R be a commutative ring with identity. Let S be a multiplicatively closed set such that $0 \notin S$. Let I be an ideal which is maximal with respect to the condition that $S \cap I = \emptyset$.

- Which of the following is necessarily true?
(1) I is a maximal ideal
(2) I is a prime ideal
(3) $I = (1)$
(4) $I = (0)$

116. Let G be a simple group of order 168. How many elements of order 7 does it have

- (1) 6 (2) 7
(3) 48 (4) 56

PART – C

117. Let a, b be positive integers with $a > b$ and $a + b = 24$. Suppose that the following congruences have a common integer solution: $2x \equiv 3a \pmod{5}$, $x \equiv 4b \pmod{5}$.



Which of the following statements are true?

- (1) $10 \leq a - b \leq 20$
- (2) $3b > a > 2b$
- (3) $a > 3b$
- (4) $a - b$ is divisible by 5

118. Consider the function $f(n) = n^5 - 2n^3 + n$, where n is a positive integer. Which of the following statements are true?

- (1) For every positive integer k , there exists a positive integer n such that $f(n)$ is divisible by 2^k .
- (2) $f(n)$ is even for every integer $n \geq 20$.
- (3) For every integer $n \geq 20$, either $f(n)$ is odd or $f(n)$ divisible by 4.
- (4) For every odd integer $n \geq 21$, $f(n)$ is divisible by 64.

119. Let $A = \mathbb{Z}[X]/(X^2 + X + 1, X^3 + 2X^2 + 2X + 6)$.

Which of the following statements are true?

- (1) A is an integral domain
- (2) A is a finite ring
- (3) A is a field
- (4) A is a product of two rings

120. Which of the following statements are necessarily true regarding a group G of order 2022?

- (1) Let g be an element of odd order in G and s_g the permutation of G given by $s_g(x) = gx$, $x \in G$. Then s_g is even permutation
- (2) The set $H = \{g \in G \mid \text{order}(g) \text{ is odd}\}$ is a normal subgroup of G
- (3) G has a normal subgroup of index 337
- (4) G has only 2 normal subgroups

121. Let p be a prime number and let $\overline{\mathbb{F}_p}$ denote an algebraic closure of the field \mathbb{F}_p . We define

$$S = \{F \subseteq \overline{\mathbb{F}_p} \mid [F : \mathbb{F}_p] < \infty\}$$

Which of the following statements are true?

- (1) S is an uncountable set
- (2) S is a countable set
- (3) For every positive integer $n > 1$, there exists a unique field $F \in S$ such that $[F : \mathbb{F}_p] = n$
- (4) Given any two fields $F_1, F_2 \in S$, either $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$

122. Which of the following are class equations for a finite group?

- (1) $1 + 3 + 3 + 3 + 3 + 13 + 13 = 39$
- (2) $1 + 1 + 2 + 2 + 2 + 2 + 2 + 2 = 14$
- (3) $1 + 3 + 3 + 7 + 7 = 21$
- (4) $1 + 1 + 1 + 2 + 5 + 5 = 15$

123. Consider $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Define a sequence of numbers F_n as follows:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ for } n = 1, 2, \dots$$

Let $p : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial of degree at most 2 such that

$$p(1) = F_1, p(3) = F_3, p(5) = F_5.$$

Which of the following statements are true?

- (1) $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$
- (2) $p(7) = 13$
- (3) $F_n = F_{n-1} + 2F_{n-2}$ for $n \geq 5$
- (4) $p(7) = 10$

JUNE 2023

PART - B

124. Let p be a prime number. Let G be a group such that for each $g \in G$ there exists an $n \in \mathbb{N}$ such that $g^{p^n} = 1$. Which of the following statements is false?

- (1) If $|G| = p^6$, then G has a subgroup of index p^2 .
- (2) If $|G| = p^6$, then G has at least five normal subgroups.
- (3) Center of G can be infinite.
- (4) There exists G with $|G| = p^6$ such that G has exactly six normal subgroups.

125. The number of solutions of the equation $x^2 = 1$ in the ring $\mathbb{Z}/105\mathbb{Z}$ is

- (1) 0
- (2) 2
- (3) 4
- (4) 8

126. Which of the following equations can occur as the class equation of a group of order 10?

- (1) $10 = 1 + 1 + \dots + 1$ (10-times)
- (2) $10 = 1 + 1 + 2 + 2 + 2 + 2$
- (3) $10 = 1 + 1 + 1 + 2 + 5$
- (4) $10 = 1 + 2 + 3 + 4$

PART - C

127. Which of the following statements are correct?



- (1) If G is a group of order 244, then G contains a unique subgroup of order 27.
- (2) If G is a group of order 1694, then G contains a unique subgroup of order 121.
- (3) There exists a group of order 154 which contains a unique subgroup of order 7.
- (4) There exists a group of order 121 which contains two subgroups of order 11.

128. Which of the following are maximal ideals of $\mathbb{Z}[X]$?

- (1) Ideal generated by 2 and $(1 + X^2)$
- (2) Ideal generated by 2 and $(1 + X + X^2)$
- (3) Ideal generated by 3 and $(1 + X^2)$
- (4) Ideal generated by 3 and $(1 + X + X^2)$

129. Let E be a finite algebraic Galois extension of F with Galois group G . Which of the following statements are true?

- (1) There is an intermediate field K with $K \neq F$ and $K \neq E$ such that K is a Galois extension of F .
- (2) If every proper intermediate field K is a Galois extension of F then G is Abelian.
- (3) If E has exactly three intermediate fields including F and E then G is Abelian.
- (4) If $[E : F] = 99$ then every intermediate field is a Galois extension of F .

130. Let $n \geq 1$ be a positive integer and S_n the symmetric group on n symbols. Let $\Delta = \{(g, g) : g \in S_n\}$. Which of the following statements are necessarily true?

- (1) The map $f: S_n \times S_n \rightarrow S_n$ given by $f(a, b) = ab$ is a group homomorphism.
- (2) Δ is a subgroup of $S_n \times S_n$.
- (3) Δ is a normal subgroup of $S_n \times S_n$.
- (4) Δ is a normal subgroup of $S_n \times S_n$, if n is a prime number.

131. Let G_1 and G_2 be two groups and $\varphi: G_1 \rightarrow G_2$ be a surjective group homomorphism. Which of the following statements are true?

- (1) If G_1 is cyclic then G_2 is cyclic
- (2) If G_1 is Abelian then G_2 is Abelian
- (3) If H is a subgroup of G_1 then $\varphi(H)$ is a subgroup of G_2

- (4) If N is a normal subgroup of G_1 then $\varphi(N)$ is a normal subgroup of G_2 .

132. Let G be a group of order 2023. Which of the following statements are true?

- (1) G is an Abelian group.
- (2) G is cyclic group.
- (3) G is a simple group.
- (4) G is not a simple group.

DECEMBER 2023

PART – B

133. Consider the field \mathbb{C} together with the Euclidean topology. Let K be a proper subfield of \mathbb{C} that is not contained in \mathbb{R} . Which one of the following statements is necessarily true?

- (1) K is dense in \mathbb{C} .
- (2) K is an algebraic extension of \mathbb{Q} .
- (3) \mathbb{C} is an algebraic extension of K .
- (4) The smallest closed subset of \mathbb{C} containing K is NOT a field.

134. Let G be any finite group. Which one of the following is necessarily true?

- (1) G is a union of proper subgroups.
- (2) G is a union of proper subgroups if $|G|$ has atleast two distinct prime divisors.
- (3) If G is abelian, then G is a union of proper subgroups.
- (4) G is a union of proper subgroups if and only if G is not cyclic.

135. Which one of the following is equal to $1^{37} + 2^{37} + \dots + 88^{37}$ in $\mathbb{Z}/89\mathbb{Z}$?

- (1) 88
- (2) -88
- (3) -2
- (4) 0

PART – C

136. Which of the following statements are true?

- (1) Let G_1 and G_2 be finite groups such that their orders $|G_1|$ and $|G_2|$ are coprime. Then any homomorphism from G_1 to G_2 is trivial.
- (2) Let G be a finite group. Let $f: G \rightarrow G$ be a group homomorphism such that f fixes more than half of the elements of G . Then $f(x) = x$ for all $x \in G$.



- (3) Let G be a finite group having exactly 3 subgroups. Then G is of order p^2 for some prime p .
- (4) Any finite abelian group G has atleast $d(|G|)$ subgroups in G , where $d(m)$ denotes the number of positive divisors of m .

137. Let $n \in \mathbb{Z}$ be such that n is congruent to 1 mod 7 and n is congruent to 4 mod 15. Which of the following statements are true?

- (1) n is congruent to 1 mod 3.
(2) n is congruent to 1 mod 35.
(3) n is congruent to 1 mod 21.
(4) n is congruent to 1 mod 5.

138. Let G be the group (under matrix multiplication) of 2×2 invertible matrices with entries from $\mathbb{Z}/9\mathbb{Z}$. Let a be the order of G . Which of the following statements are true?

- (1) a is divisible by 3^4 .
(2) a is divisible by 2^4 .
(3) a is not divisible by 48.
(4) a is divisible by 3^6 .

139. Let $R = \mathbb{Z}[X]/(X^2 + 1)$ and $\psi : \mathbb{Z}[X] \rightarrow R$ be the natural quotient map. Which of the following statements are true?

- (1) R is isomorphic to a subring of \mathbb{C}
(2) For any prime number $p \in \mathbb{Z}$, the ideal generated by $\psi(p)$ is a proper ideal of R .
(3) R has infinitely many prime ideals.
(4) The ideal generated by $\psi(X)$ is a prime ideal in R .

140. Let $f(X) = X^2 + X + 1$ and $g(X) = X^2 + X - 2$ be polynomials in $\mathbb{Z}[X]$. Which of the following statements are true?

- (1) For all prime numbers p , $f(X) \bmod p$ is irreducible in $(\mathbb{Z}/p\mathbb{Z})[X]$
(2) There exists a prime number p such that $g(X) \bmod p$ is irreducible in $(\mathbb{Z}/p\mathbb{Z})[X]$
(3) $g(X)$ is irreducible in $\mathbb{Q}[X]$
(4) $f(X)$ is irreducible in $\mathbb{Q}[X]$

141. Let $f(X) = X^3 - 2 \in \mathbb{Q}[X]$ and let $K \subset \mathbb{C}$ be the splitting field of $f(X)$ over \mathbb{Q} . Let $\omega =$

$e^{2\pi i/3}$. Which of the following statements are true?

- (1) The Galois group of K over \mathbb{Q} is the symmetric group S_3 .
(2) The Galois group of K over $\mathbb{Q}(\omega)$ is the symmetric group S_3 .
(3) The Galois group of K over \mathbb{Q} is $\mathbb{Z}/3\mathbb{Z}$.
(4) The Galois group of K over $\mathbb{Q}(\omega)$ is $\mathbb{Z}/3\mathbb{Z}$.



ANSWERS

- | | | |
|----------------|----------------|---------------|
| 1. (2) | 2. (2) | 3. (3) |
| 4. (1) | 5. (4) | 6. (2,3) |
| 7. (1,3) | 8. (2,3,4) | 9. (1,2,3,4) |
| 10. (3) | 11. (2) | 12. (3) |
| 13. (4) | 14. (1,2,3) | 15. (2,3) |
| 16. (1,2,3,4) | 17. (1,3) | 18. (1,3) |
| 19. (2,4) | 20. (1,2,4) | 21. (3) |
| 22. (3) | 23. (4) | 24. (2) |
| 25. (1) | 26. (3,4) | 27. (1,4) |
| 28. (3,4) | 29. (2,4) | 30. (4) |
| 31. (1,4) | 32. (3) | 33. (1) |
| 34. (1) | 35. (2) | 36. (1,4) |
| 37. (1,2,3,4) | 38. (1,4) | 39. (1,2,4) |
| 40. (1,2) | 41. (4) | 42. (3) |
| 43. (1,2,3,4) | 44. (2) | 45. (1,3,4) |
| 46. (1,2,3) | 47. (1,2,3,4) | 48. (2,4) |
| 49. (4) | 50. (1,2,4) | 51. (2,4) |
| 52. (1,4) | 53. (1,4) | 54. (2,4) |
| 55. (1,4) | 56. (2) | 57. (2) |
| 58. (1) | 59. (3) | 60. (1,4) |
| 61. (1,2,3,4) | 62. (3,4) | 63. (1) |
| 64. (3) | 65. (4) | 66. (2) |
| 67. (1) | 68. (1,4) | 69. (1) |
| 70. (2,3) | 71. (1) | 72. (3) |
| 73. (1) | 74. (1) | 75. (1,2,3,4) |
| 76. (2) | 77. (4) | 78. |
| 79. (3) | 80. (4) | 81. (4) |
| 82. (2,3,4) | 83. (1,4) | 84. (1,3) |
| 85. (1,2,3) | 86. (2,3) | 87. (1,2,4) |
| 88. (1) | 89. (3) | 90. (2) |
| 91. (1,4) | 92. (1,2) | 93. (1,2,3) |
| 94. (1) | 95. (1,2) | 96. (3) |
| 97. (3) | 98. (3) | 99. (4) |
| 100. (2,3) | 101. (1,3) | 102. (1,2) |
| 103. (1,2,3,4) | 104. (1,2,3) | 105. (1,2) |
| 106. (2) | 107. (2) | 108. (2) |
| 109. (2,4) | 110. (1,3) | 111. (2,4) |
| 112. () | 113. (3) | 114. (4) |
| 115. (2) | 116. (3) | 117. (1,4) |
| 118. (1,2,4) | 119. (1,2,3) | 120. (1,2) |
| 121. (2,3) | 122. (1,3) | 123. (1,4) |
| 124. (4) | 125. (4) | 126. (1) |
| 127. (2,3,4) | 128. (2,3) | 129. (3,4) |
| 130. (2) | 131. (1,2,3,4) | 132. (1,4) |
| 133. (1) | 134. (4) | 135. (4) |
| 136. (1,2,3,4) | 137. (1,3) | 138. (1,2) |
| 139. (1,2,3) | 140. (4) | 141. (1,4) |